Maximum principle for the fractional diffusion equations and its applications

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Outline of the talk:

- General fractional derivative and its properties
- Maximum principle for a time-fractional diffusion equation with the general fractional derivative
- Maximum principle for a time-fractional diffusion equation with the Caputo fractional derivative and its applications
- Non-negativity of solutions to an abstract space- and time-fractional evolution equation in the Hilbert space
- Survey of other results

What is a maximum principle?

Maximum principle:
A function satisfies a differential inequality or equation in a domain $D \Rightarrow$
It achieves its maximum on the boundary of D.

A very elementary example:
$f''(x) > 0,\ x \in]a, b[$ and $f \in C([a, b]) \Rightarrow$
f achieves its maximum value at one of the endpoints of the interval.

Other examples:

- Maximum principles for ordinary differential equations and inequalities
- Maximum principles for partial differential equations and inequalities
- Recently: maximum principles for fractional PDEs

Applications:

- Analysis of direct and inverse problems for ODEs and PDEs
- Construction of suitable numerical algorithms
- Optimisation ???
- ...

General fractional derivatives

Let k be a nonnegative locally integrable function on $\mathbb{R}_+$.

The general fractional derivative with the kernel k:

$$(\mathbb{D}_{(k)} f)(t) = \frac{d}{dt}(k * f)(t) = \frac{d}{dt}\int_0^t k(t-\tau)f(\tau)\,d\tau.$$

The regularized general fractional derivative with the kernel k:

$$({}_*\mathbb{D}_{(k)} f)(t) = \frac{d}{dt}\int_0^t k(t-\tau)(f(\tau)-f(0))\,d\tau = (\mathbb{D}_{(k)} f)(t) - k(t)f(0).$$

For $f' \in L_1^{loc}(\mathbb{R}_+)$, we get

$$({}_*\mathbb{D}_{(k)} f)(t) = \int_0^t k(t-\tau)f'(\tau)\,d\tau.$$

Particular cases

1) The conventional Riemann-Liouville and Caputo fractional derivatives:

$$k(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}, \ 0 < \alpha < 1.$$

$$(\mathbb{D}_{(k)} f)(t) = (\mathbb{D}^{\alpha} f)(t) = \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f(\tau) \, d\tau \right),$$

$$({}_*\mathbb{D}_{(k)} f)(t) = (\mathbb{D}^{\alpha}_* f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) \, d\tau.$$

Particular cases

2) The multi-term Riemann-Liouville and Caputo fractional derivatives:

$$k(\tau) = \sum_{k=1}^{n} a_k \frac{\tau^{-\alpha_k}}{\Gamma(1-\alpha_k)}, \quad 0 < \alpha_1 < \cdots < \alpha_n < 1.$$

$$(\mathbb{D}_{(k)} f)(t) = \sum_{k=1}^{n} a_k (\mathbb{D}^{\alpha_k} f)(t),$$

$$({}_*\mathbb{D}_{(k)} f)(t) = \sum_{k=1}^{n} a_k (\mathbb{D}_*^{\alpha_k} f)(t),$$

Particular cases

3) The Riemann-Liouville and Caputo fractional derivatives of distributed order:

$$k(\tau) = \int_0^1 \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\, d\rho(\alpha),$$

where ρ is a Borel measure on $[0, 1]$.

$$(\mathbb{D}_{(k)}\, f) = \int_0^1 (\mathbb{D}^{\alpha} f)(t)\, d\rho(\alpha),$$

$$({}_*\mathbb{D}_{(k)}\, f) = \int_0^1 (\mathbb{D}^{\alpha}_* f)(t)\, d\rho(\alpha).$$

Kochubei conditions on the kernel function k

A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes. *Integr. Equa. Operator Theory* **71** (2011), 583–600.

K1) The Laplace transform $\tilde{k}$ of k,

$$\tilde{k}(p) = \int_0^\infty k(t)\, e^{-pt}\, dt,$$

exists for all $p > 0$,

K2) $\tilde{k}(p)$ is a Stiltjes function (it possesses a special integral representation),

K3) $\tilde{k}(p) \to 0$ and $p\tilde{k}(p) \to \infty$ as $p \to \infty$,

K4) $\tilde{k}(p) \to \infty$ and $p\tilde{k}(p) \to 0$ as $p \to 0$.

Basic properties of the general fractional derivatives

(A) For any $\lambda > 0$, the initial value problem for the fractional relaxation equation

$$({}_*\mathbb{D}_{(k)} f)(t) = -\lambda f(t), \quad t > 0, \quad u(0) = 1$$

has a unique solution $u_\lambda = u_\lambda(t)$ that belongs to the class $C^\infty(\mathbb{R}_+)$ and is a completely monotone function.

(B) There exists a completely monotone function $\kappa = \kappa(t)$ with the property (Sonin condition)

$$(k * \kappa)(t) = \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1, \ t > 0.$$

(C) For $f \in L_1^{loc}(\mathbb{R}_+)$, the relations

$$(\mathbb{D}_{(k)}\, \mathbb{I}_{(k)} f)(t) = f(t), \quad ({}_*\mathbb{D}_{(k)}\, \mathbb{I}_{(k)} f)(t) = f(t)$$

hold true, where the general fractional integral $\mathbb{I}_{(k)}$ is defined by the formula

$$(\mathbb{I}_{(k)} f)(t) = (\kappa * f)(\tau) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau.$$

General time-fractional diffusion equation

Main object of todays talk:

The general time-fractional diffusion equation:

$$({}_*\mathbb{D}_{(k)}\, u(x,\cdot))(t) = D_2(u) + D_1(u) - q(x)u(x,t) + F(x,t), \quad (x,t) \in \Omega \times (0,T],$$

where $q \in C(\bar{\Omega})$,

$$D_2(u) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad D_1(u) = \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}.$$

and D_2 is a uniformly elliptic differential operator.

Cauchy problem

A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes. *Integr. Equa. Operator Theory* **71** (2011), 583–600.

The general time-fractional diffusion equation

$$({}_*\mathbb{D}_{(k)}\, u(x,\cdot))(t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T]$$

with the initial condition

$$u(x,0) = u_0(x), x \in \mathbb{R}^n.$$

1) The Cauchy problem has a unique appropriately defined solution for a bounded globally Hölder continuous initial value u_0.

2) The fundamental solution to the Cauchy problem can be interpreted as a probability density function evolving in time and thus the general time-fractional diffusion equation describes a kind of (anomalous) diffusion.

Initial-boundary-value problem

Yu. Luchko and M. Yamamoto, General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.* **19** (2016), 676–695.

Let Ω be an open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (for example, of C^2 class) and $T > 0$.

The general time-fractional diffusion equation

$$({}_*\mathbb{D}_{(k)}\, u(x,\cdot))(t) = D_2(u) + D_1(u) - q(x)u(x,t) + F(x,t), \;\; (x,t) \in \Omega \times (0,T],$$

with the initial condition

$$u(x,t)\big|_{t=0} = u_0(x), \;\; x \in \bar{\Omega}$$

and the Dirichlet boundary condition

$$u(x,t)\big|_{(x,t)\in\partial\Omega\times(0,T]} = v(x,t), \;\; (x,t) \in \partial\Omega \times (0,T].$$

Initial-boundary-value problem

Yu. Luchko and M. Yamamoto, General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.* **19** (2016), 676–695.

Main results:

1) Uniqueness of solution both in the strong and in the weak senses (Estimates of the general fractional derivatives at their maximum points –> Maximum principle for the general fractional diffusion equation –> A priory norm estimates of solutions –> Uniqueness of solutions).

2) Existence of solution in the weak sense (Separation of variables –> Formal solutions in form of generalized Fourier series –> Convergence analysis of the formal solutions).

Estimates of the general fractional derivatives

Let the conditions

L1) $k \in C^1(\mathbb{R}_+) \cap L_1^{loc}(\mathbb{R}_+)$,

L2) $k(\tau) > 0$ and $k'(\tau) < 0$ for $\tau > 0$,

L3) $k(\tau) = o(\tau^{-1}),\ \tau \to 0$.
be fulfilled.

Let a function $f \in C([0, T])$ attain its maximum over the interval $[0, T]$ at the point $t_0,\ t_0 \in (0, T]$, and $f' \in C(0, T] \cap L_1(0, T)$. Then the following inequalities hold true:

$$(\mathbb{D}_{(k)} f)(t_0) \geq k(t_0) f(t_0),$$

$$({}_*\mathbb{D}_{(k)} f)(t_0) \geq k(t_0)(f(t_0) - f(0)) \geq 0.$$

Sketch of the proof

Under the conditions posed on the functions f and k, the general fractional derivative $\mathbb{D}_{(k)}$ can be represented as follows:

$$(\mathbb{D}_{(k)} f)(t) = I(t) + f'(t) \int_0^t k(t-\tau)\, d\tau + k(t) f(t),$$

where

$$I(t) = \frac{d}{dt} \int_0^t k(t-\tau)(f(\tau) - f(t))\, d\tau.$$

Due to the relation

$$\lim_{\tau \to t} k(t-\tau)(f(\tau) - f(t)) = - \lim_{\tau \to t} k(t-\tau) f'(\zeta)(t-\tau) = 0,$$

we get the formula

$$I(t) = \int_0^t k'(t-\tau)(f(\tau) - f(t))\, d\tau - f'(t) \int_0^t k(t-\tau)\, d\tau.$$

Sketch of the proof

Substituting the last formula into the first one, we arrive at the following representation

$$(\mathbb{D}_{(k)}\, f)(t) = \int_0^t k'(t-\tau)(f(\tau)-f(t))\, d\tau \;+\; k(t)f(t)$$

that is valid for all $t \in (0, T]$. At the point $t = t_0$, the function f attends its maximum ($f(\tau) - f(t_0) \leq 0$) and the condition L2) implicates that $k'(\tau) < 0,\ \tau > 0$. We thus obtain the inequality

$$(\mathbb{D}_{(k)}\, f)(t_0) = \int_0^{t_0} k'(t_0-\tau)(f(\tau)-f(t_0))\, d\tau \;+\; k(t_0)f(t_0) \geq k(t_0)f(t_0).$$

For the regularized general fractional derivative, we readily arrive at the estimate

$$({}_*\mathbb{D}_{(k)}f)(t_0) = (\mathbb{D}_{(k)}f)(t_0) - k(t_0)f(0) \geq k(t_0)f(t_0) - k(t_0)f(0) \geq 0$$

and the proof is completed.

Particular cases

For the Riemann-Liouville and Caputo fractional derivatives the derived inequalities take the form

$$(\mathbb{D}^{\alpha} f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} f(t_0)$$

and

$$(\mathbb{D}_*^{\alpha} f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} (f(t_0) - f(0)) \geq 0.$$

Initial-boundary-value problem for the general time-fractional diffusion equation

The general time-fractional diffusion equation

$$({}_*\mathbb{D}_{(k)}\, u(x,\cdot))(t) = D_2(u) + D_1(u) - q(x)u(x,t) + F(x,t),\ \ (x,t) \in \Omega \times (0,T],$$

with the initial condition

$$u(x,t)\big|_{t=0} = u_0(x),\ \ x \in \bar{\Omega}$$

and the Dirichlet boundary condition

$$u(x,t)\big|_{(x,t)\in\partial\Omega\times(0,T]} = v(x,t),\ \ (x,t) \in \partial\Omega \times (0,T].$$

Maximum principle

Let us define the operator

$$\mathbb{P}_{(k)}(u) := ({}_*\mathbb{D}_{(k)}\, u(x,\cdot))(t) - D_2(u) - D_1(u) + q(x)u(x,t).$$

Let the conditions L1)-L3) be fulfilled and a function u satisfy the inequality

$$\mathbb{P}_{(k)}(u) \le 0,\ (x,t) \in \Omega \times (0,T].$$

Then the following maximum principle holds true:

$$\max_{(x,t)\in\ \bar{\Omega}\times[0,T]} u(x,t) \le \max\{\max_{x\in\bar{\Omega}} u(x,0), \max_{(x,t)\in\ \partial\Omega\times[0,T]} u(x,t),\ 0\}.$$

A priori norm estimates

Let the conditions K1)-K4) and L1)-L3) be fulfilled and u be a solution of the initial-boundary-value problem for the general fractional diffusion equation.

Then the following estimate of the uniform solution norm holds true:

$$\|u\|_{C(\bar{\Omega}\times[0,T])} \leq \max\{M_0,\ M_1\} + M\, f(T),$$

where

$$M_0 = \|u_0\|_{C(\bar{\Omega})},\ M_1 = \|v\|_{C(\partial\Omega\times[0,T])},\ M = \|F\|_{C(\Omega\times[0,T])},$$

and

$$f(t) = \int_0^t \kappa(\tau)\, d\tau, \quad \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1,\ t > 0.$$

Uniqueness of the solution

The initial-boundary-value problem for the general fractional diffusion equation possesses at most one solution.

This solution continuously depends on the problem data in the sense that if u and $\tilde{u}$ solutions to the problems with the sources functions F and $\tilde{F}$ and the initial and boundary conditions u_0 and $\tilde{u}_0$ and v and $\tilde{v}$, respectively, and

$$\|F - \tilde{F}\|_{C(\bar{\Omega}\times[0,T])} \leq \epsilon,$$

$$\|u_0 - \tilde{u}_0\|_{C(\bar{\Omega})} \leq \epsilon_0, \ \|v - \tilde{v}\|_{C(\partial\Omega\times[0,T])} \leq \epsilon_1,$$

then the following norm estimate holds true:

$$\|u - \tilde{u}\|_{C(\bar{\Omega}\times[0,T])} \leq \max\{\epsilon_0, \epsilon_1\} + \epsilon\, f(T)$$

with

$$f(t) = \int_0^t \kappa(\tau)\, d\tau, \quad \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1, \ t > 0.$$

Time-fractional diffusion equation with the Caputo derivative

Initial-boundary-value problem:

$$\partial_t^\alpha u(x,t) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u(x,t)) + c(x)u(x,t) + F(x,t),\ x \in \Omega \subset \mathbb{R}^n,\ t > 0$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0,$$

$$u(x,0) = a(x), \qquad x \in \Omega$$

with $0 < \alpha < 1$ and in a bounded domain Ω with a smooth boundary $\partial\Omega$. In what follows, we always suppose that the spatial differential operator is a uniformly elliptic one.

Weak solution in the fractional Sobolev space

For $u \in C^1[0,T]$, the Caputo fractional derivative is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s) ds, \quad x \in \Omega.$$

This fractional derivative ∂_t^α can be extended to an operator defined on the closure $H_\alpha(0,T)$ of ${}_0C^1[0,T] := \{u \in C^1[0,T];\ u(0)=0\}$ in the fractional Sobolev space $H^\alpha(\Omega)$.

In what follows, we interpret $\partial_t^\alpha u$ as this extension with the domain $H_\alpha(0,T)$.

Thus we interpret the problem under consideration as the fractional diffusion equation subject to the inclusions

$$\begin{cases} u(\cdot,t) \in H_0^1(\Omega), \quad t>0, \\ u(x,\cdot) - a(x) \in H_\alpha(0,T), \quad x \in \Omega. \end{cases}$$

Results regarding the maximum principle

Yu. Luchko, Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* **351** (2009), 218–223.

Maximum principle for the strong solution under the assumption

$$c(x) \leq 0, \quad x \in \overline{\Omega}.$$

Yu. Luchko, M. Yamamoto, On the maximum principle for a time-fractional diffusion equation. *Fract. Calc. Appl. Anal.* **20** (2017), 1131–1145.

Maximum principle for the weak solution in the case $c \in C(\overline{\Omega})$ without the non-negativity condition $c(x) \leq 0, \ x \in \overline{\Omega}$.

Consequences from the maximum principle

Let us now denote the solution to the initial-boundary value problem for the fractional diffusion equation by $u_{a,F}$.
1) Non-negativity property:

Let $a \in L^2(\Omega)$ and $F \in L^2(\Omega \times (0,T))$. If $F(x,t) \geq 0$ a.e. (almost everywhere) in $\Omega \times (0,T)$ and $a(x) \geq 0$ a.e. in Ω, then $u_{a,F}(x,t) \geq 0$ a.e. in $\Omega \times (0,T)$.

2) Comparison property:

Let $a_1, a_2 \in L^2(\Omega)$ and $F_1, F_2 \in L^2(\Omega \times (0,T))$ satisfy the inequalities $a_1(x) \geq a_2(x)$ a.e. in Ω and $F_1(x,t) \geq F_2(x,t)$ a.e. in $\Omega \times (0,T)$, respectively. Then $u_{a_1,F_1}(x,t) \geq u_{a_2,F_2}(x,t)$ a.e. in $\Omega \times (0,T)$.

Positivity property

Yu. Luchko and M. Yamamoto, *Maximum principle for the time-fractional PDEs.* Chapter in: A. Kochubei, Yu. Luchko (Eds.), Handbook of Fractional Calculus with Applications. Vol.2: Fractional Differential Equations, Walter de Gruyter, Berlin/Boston, 2019, pp. 299-326

Conditions:
1) the initial condition $a \in L^2(\Omega)$, $a \geq 0$, $a \not\equiv 0$ a.e. in Ω,
2) the weak solution u belongs to $C((0,T]; C(\overline{\Omega}))$,
3) the source function is identically equal to zero, i.e.,
$F(x,t) \equiv 0, \ x \in \Omega, \ t > 0.$

Then the weak solution u is strictly positive:

$$u(x,t) > 0, \quad x \in \Omega, \, 0 < t \leq T.$$

Abstract time- and space-fractional diffusion equation

Let X be a Hilbert space over $\mathbb{R}$ with the scalar product $(\cdot, \cdot)$. For $0 < \alpha, \beta < 1$, we consider the following evolution equation in the Hilbert space X with the time-fractional Caputo derivative D_t^α:

$$D_t^\alpha u(t) = -(-A)^\beta u \quad \text{in } X,\ t > 0$$

along with the initial condition

$$u(0) = a \in X.$$

Assumptions: the operator $-A$ is self-adjoint and positive definite $(\sigma(-A) \subset [0, +\infty))$.

We note that $u(\cdot, t) := u(t) \in D((-A)^\beta)$ for $t > 0$ and so a boundary condition is incorporated into the equation.

Fractional Laplacian or the Riesz fractional derivative

For $A = \Delta$, there are many (at least 10) different interpretations of $-(-A)^{\beta}$ like e.g.

a) pseudo-differential operator with the symbol $-|\kappa|^{2\beta}$

$$\left(\mathcal{F} - (-\Delta)^{\beta} f\right)(\kappa) = -|\kappa|^{2\beta} (\mathcal{F} f)(\kappa),$$

b) hypersingular integral (inversion of the Riesz potential)

$$-(-\Delta)^{\beta} f(\mathrm{x}) = \frac{1}{d_{n,m}(\beta)} \int_{\mathbb{R}^n} \frac{\left(\Delta_h^m f\right)(\mathrm{x})}{|h|^{n+2\beta}}\, dh$$

with the suitably defined finite differences operator $\left(\Delta_h^m f\right)(\mathrm{x})$ and the normalization constant $d_{n,m}(\beta)$.

Non-negativity property

For $0 < \alpha, \beta < 1$, let us denote a solution to the abstract time- and space-fractional diffusion equation by $u_{\alpha,\beta}(t)$.

If $a \geq 0$ in Ω, then $u_{\alpha,\beta}(\cdot, t) \geq 0$ in Ω for $t \geq 0$.

Yu. Luchko, M. Yamamoto, *A Survey on the Recent Results Regarding Maximum Principles for the Time-Fractional Diffusion Equations*, Chapter in: Bhalekar, Sachin (Ed.), Frontiers in Fractional Calculus, Bentham Science Publishers, UAE, 2018, pp. 33-69.

Sketch of the proof

The main idea is first to prove non-negativity of $u_{\alpha,\beta}$ in the case $\alpha = 1$ (space-fractional equation) and then to extend this result for the general case.

We start with the following important result:

$u_{1,\beta}(\cdot, t) \geq 0$ in Ω for $t \geq 0$ if $a \geq 0$ in Ω, where $u_{1,\beta}(\cdot, t)$ is solution to the evolution equation

$$\frac{\partial}{\partial t} u(t) = -(-A)^{\beta} u \quad \text{in } X,\ t > 0$$

along with the initial condition

$$u(0) = a \in X.$$

Sketch of the proof

Ingredients for the proof:
1) Integral representation

$$((-A)^{\beta}+1)^{-1}a=\frac{\sin\pi\beta}{\pi}\int_0^{\infty}\frac{\mu^{\beta}(-A+\mu)^{-1}a}{\mu^{2\beta}+2\mu^{\beta}\cos\pi\beta+1}d\mu,\quad a\in X$$

2) $(-A+\mu)^{-1}a\geq 0$ for $\mu\geq 0$ and $a\geq 0$ in Ω.
3)$\mu^{2\beta}+2\mu^{\beta}\cos\pi\beta+1>0$ for $\mu\geq 0$ and $0<\beta<1$.

Hence

$$(1+(-A)^{\beta})^{-1}a\geq 0\quad \text{if}\ \ a(x)\geq 0,\ x\in\Omega.$$

Then

$$u_{1,\beta}(\cdot,t)=e^{-(-A)^{\beta}t}a=\lim_{k\to\infty}\left(1+\frac{t}{k}(-A)^{\beta}\right)^{-k}a\geq 0.$$

Sketch of the proof

Let $0 < \alpha, \beta < 1$. Then

$$u_{\alpha,\beta}(x,t) = \int_0^\infty \Phi_\alpha(\eta) u_{1,\beta}(x, t^\alpha \eta) d\eta, \quad x \in \Omega,\ t > 0$$

with

$$\Phi_\alpha(\eta) = \sum_{k=0}^\infty \frac{(-\eta)^k}{k! \Gamma(-\alpha k + 1 - \alpha)}$$

being a particular case of the Wright function (also known as the Mainardi function).

Because $u_{1,\beta}(x,t) \geq 0$ for $x \in \Omega$ and $t \geq 0$ for $a \geq 0$ and

$$\Phi_\alpha(\eta) \geq 0, \quad \eta > 0$$

we get the inequality

$$u_{\alpha,\beta}(x,t) \geq 0,\ x \in \Omega,\ t \geq 0.$$

Some of other results

A. Alsaedi, B. Ahmad and M. Kirane, Maximum principle for certain generalized time and space fractional diffusion equations. *Quart. Appl. Math.* **73** (2015), 163–175.

M. Al-Refai, Yu. Luchko, Maximum principles for the fractional diffusion equations with the Riemann-Liouville fractional derivative and their applications. *Fract. Calc. Appl. Anal.* **17** (2014), 483–498.

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M. Al-Refai, Yu. Luchko, Analysis of fractional diffusion equations of distributed order: Maximum principles and their applications. *Analysis* **36** (2016), 123–133.

Some of other results

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Some of other results

Yu. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation. *J. Math. Anal. Appl.* **374** (2011), 538–548.

Yu. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.* **12** (2009), 409–422.

H. Ye, F. Liu, V. Anh, I. Turner, Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations. *Appl. Math. Comput.* **227** (2014), 531–540.

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Thanks for your attention!

$$\max_{(x,t)\in\,\bar{\Omega}\times[0,T]} u(x,t) \le \max\{\max_{x\in\bar{\Omega}} u(x,0), \max_{(x,t)\in\,\partial\Omega\times[0,T]} u(x,t),\ 0\}$$

Questions and comments are welcome!