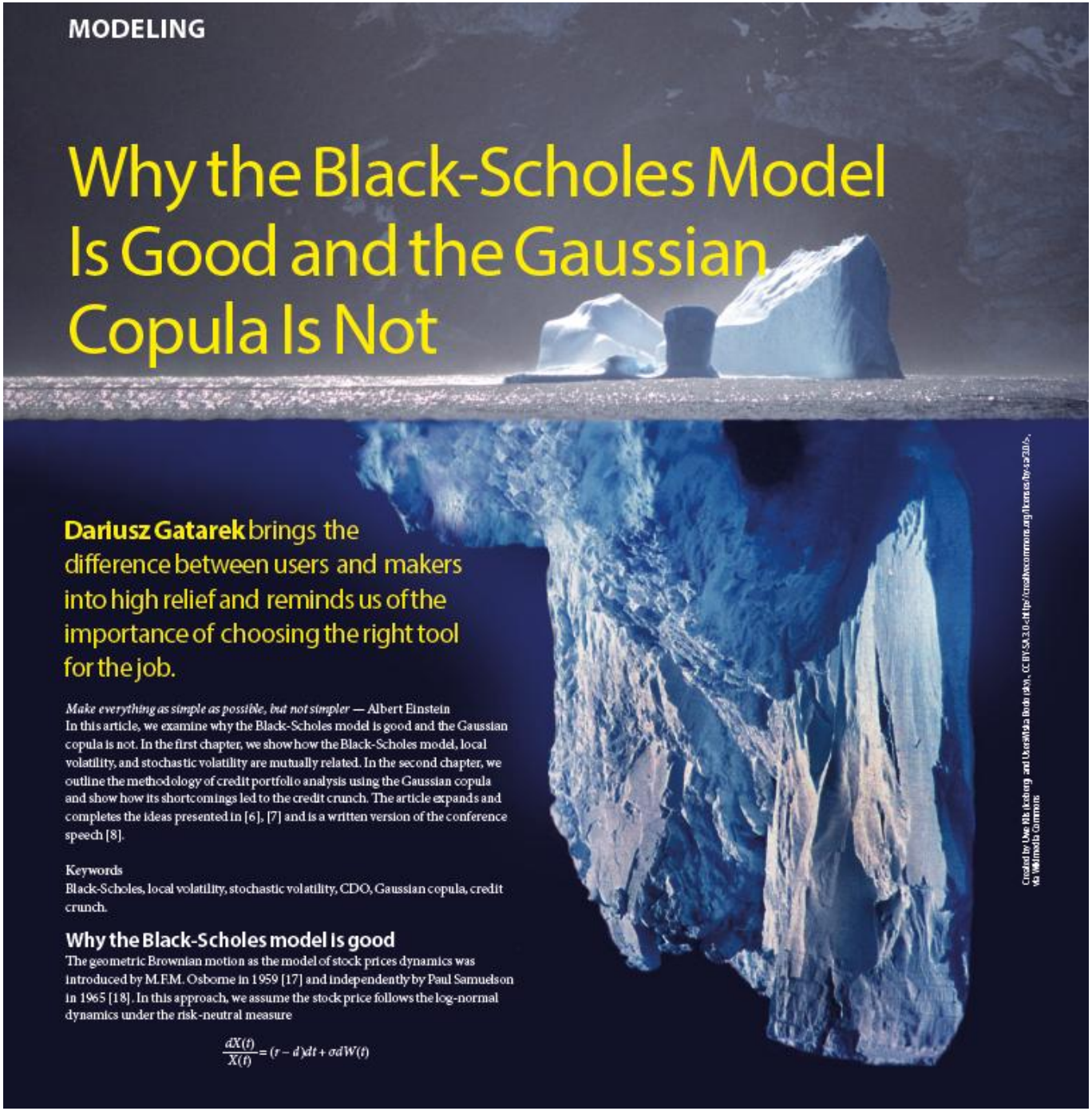

AND NOW FOR
SOMETHING
COMPLETELY
DIFFERENT

Why the Black-Scholes Model Is Good and the Gaussian Copula Is Not



Dariusz Gatarek brings the difference between users and makers into high relief and reminds us of the importance of choosing the right tool for the job.

Make everything as simple as possible, but not simpler — Albert Einstein
In this article, we examine why the Black-Scholes model is good and the Gaussian copula is not. In the first chapter, we show how the Black-Scholes model, local volatility, and stochastic volatility are mutually related. In the second chapter, we outline the methodology of credit portfolio analysis using the Gaussian copula and show how its shortcomings led to the credit crunch. The article expands and completes the ideas presented in [6], [7] and is a written version of the conference speech [8].

Keywords
Black-Scholes, local volatility, stochastic volatility, CDO, Gaussian copula, credit crunch.

Why the Black-Scholes model is good

The geometric Brownian motion as the model of stock prices dynamics was introduced by M.F.M. Osborne in 1959 [17] and independently by Paul Samuelson in 1965 [18]. In this approach, we assume the stock price follows the log-normal dynamics under the risk-neutral measure

$$\frac{dX(t)}{X(t)} = (r - d)dt + \sigma dW(t)$$

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Mathematics cannot fill your life. Mathematical ignorance can.

Hugo Steinhaus



Make everything as simple as possible, but not simpler.

Albert Einstein

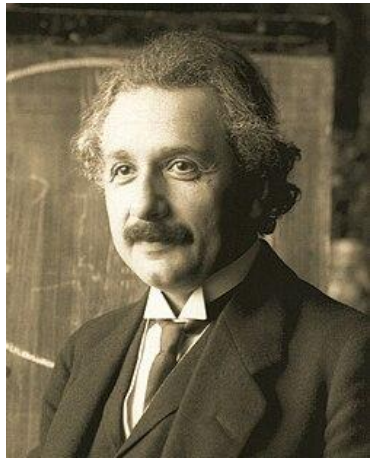


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Black-Scholes formula

Assume the stock price follows the well known log-normal dynamics under the risk neutral measure

$$\frac{dX(t)}{X(t)} = (r - d)dt + \sigma dW(t)$$

where $W(t)$ is the Wiener process, μ is the expected rate of return of the stock, r is the short rate of the economy, d is the dividend rate and σ the volatility.



Black-Scholes equation

Assume the following:

1. The interest rate r is deterministic and constant.
2. Stock prices are lognormally distributed.
3. The volatility sigma is constant.
4. The dividend rate d is deterministic and constant.

Define

$$Call(x,t,T,K) = C(x,t) = e^{-rT}E((X(T)-K)^{+}|X(t)=x).$$

Black-Scholes equation

$$\frac{\partial C}{\partial t} + (r-d)x\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} = rC,$$

$$C(x,T) = (x-K)^{+}.$$

Black-Scholes formula

Then

$$Call(X(0), 0, r, \sigma, d, T, K) = X(0)e^{-dT}\Phi(h) - Ke^{-rT}\Phi\left(h - \sigma\sqrt{T}\right),$$

where

$$h = \frac{\ln X(0) - \ln K + \left(r - d + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and Φ is the cumulative normal distribution.

Volatility smile

How Black and Scholes formula is used in practice

$$Call(X(0), T, K) = X(0)e^{-dT}\Phi(h) - Ke^{-rT}\Phi\left(h - \sigma(T, K)\sqrt{T}\right),$$

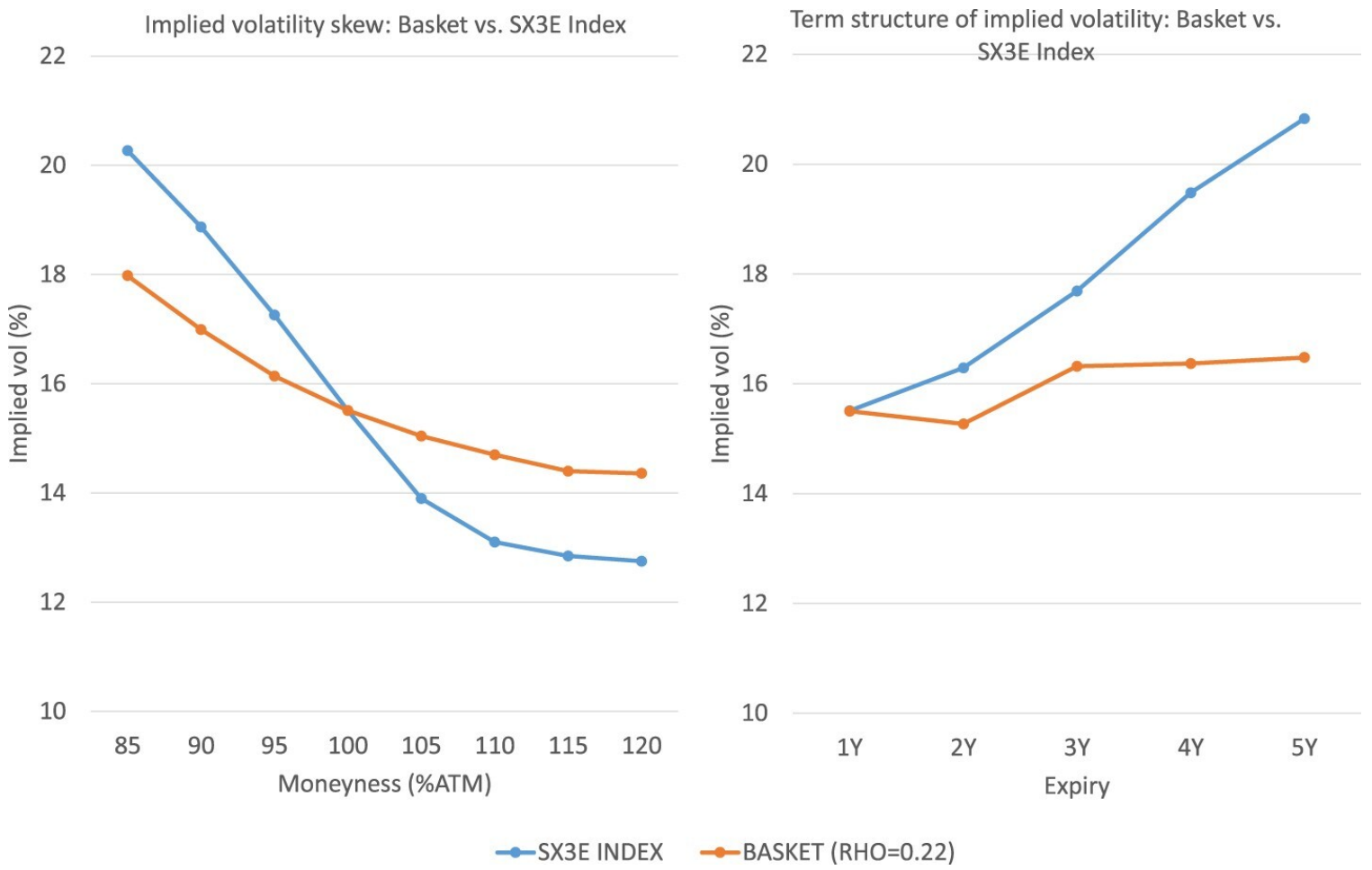
where

$$h = \frac{\ln X(0) - \ln K + \left(r - d + \frac{1}{2}\sigma^2\,(T, K)\right)T}{\sigma(T, K)\sqrt{T}}.$$

Is it mathematically consistent?

1. No, it isn't.
2. Yes, it is.

Volatility smile



Gatheral formula

Define $v(T,K)$ by the following formula:

$$\frac{\partial w}{\partial T} = v^2\left\{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(\frac{y^2}{w^2} - \frac{1}{4} - \frac{1}{w}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial^2 y}\right\},$$

Where $y = \ln K/X(0) + (d-r)T$ and $w = \sigma^2(T,K)T$.

Let

$$\frac{dY(t)}{Y(t)} = (r-d)dt + v\big(t, Y(t)\big)dW(t).$$

Then

$$Call(X(0),T,K) = e^{-rT}E(Y(T)-K)^+.$$

Gyöngy theorem

Let $S(t)$ be a diffusion process following the equation:

$$\frac{dS(t)}{S(t)} = (r-d)dt + \sigma(t)dW(t),$$

Where the volatility $\sigma(t)$ is a stochastic process. There exists a Markov diffusion $Y(t)$ with volatility $v(t,x)$

$$v^2(t,x) = E(\sigma^2(t)|X(t) = x)$$

Following the equation

$$\frac{dY(t)}{Y(t)} = (r-d)dt + v(t,Y(t))dW(t).$$

Both processes $S(t)$ and $Y(t)$ have the same marginal distributions, hence option prices coincide.

Stochastic interest rates

Denote by $B(t,T)$ be discount factors on the period $[t,T]$ – understood as value at time *t* of an obligation to pay \$1 at time *T*. Payment of this dollar is certain; there is no credit risk involved. This obligation is also called a zero-coupon bond. Let $r(t)$ be the short rate defined as $r(t)dt = 1 - B(t, t+dt)$.

According to Harrison and Pliska we say that the bond market is arbitrage free if

$$M(t,T) = \frac{B(t,T)}{B(0,T)} \exp\left(- \int_0^t r(s)ds \right)$$

is a positive continuous martingale.

Define the forward measure E_T by $E_T \xi M(T,T)$ for any random variable ξ. The option price is then equal

$$Call = B(0,T)E_T(X(T) - K)^+ = e^{-rT}E_T(X(T) - K)^+.$$

Black formula for interest rates

Let δ be accrual period for both interest rates and swaps. Define successive grid points as $T_{i+1} = T_i + \delta$ for a certain initial $T = T_0$. The forward rates are defined as

$$\delta L_n(t) + 1 = \frac{B(t, T_{n-1})}{B(t, T_n)}.$$

The market convention of pricing interest rate options is the Black-Scholes formula, in this particular case called the Black futures formula:

$$Caplet = \delta B(0, T_n)\left(L_n(0)\Phi(h) - K\Phi\left(h - \sigma\sqrt{T_{n-1}}\right)\right),$$

where

$$h = \frac{\ln L_n(0) - \ln K + \frac{1}{2}\sigma^2 T_{n-1}}{\sigma\sqrt{T_{n-1}}}$$

and σ is the volatility parameter.

Black formula for interest rates

For swaptions the pricing convention is even more interesting. Forward swap rates are defined as

$$S_{nN}(t) = \frac{\sum_{i=n+1}^{N} B(t,T_i)L_i(t)}{\sum_{i=n+1}^{N} B(t,T_i)} = \frac{B(t,T_n) - B(t,T_N)}{\delta \sum_{i=n+1}^{N} B(t,T_i)}.$$

The Black futures formula for swaptions looks as follows:

$$Swaption_{nN} = \delta \sum_{i=n+1}^{N} B(0,T_i)\left(S_{nN}(0)\Phi(h) - K\Phi\left(h - \sigma\sqrt{T_n}\right)\right),$$

where

$$h = \frac{\ln S_{nN}(0) - \ln K + \frac{1}{2}\sigma^2 T_n}{\sigma\sqrt{T_n}}.$$

Why models are so important

- **Consistent** pricing of all instruments
- Proper risk management

Model → Calibration → Pricing

Why models are so important



Black-Scholes

Local volatility

Stochastic volatility

Heath-Jarrow-Morton

Collateralized debt obligation

Let the credit portfolio consist of N obligors with default times $\tau_1, \tau_2, \tau_3, \dots, \tau_N$ with respective distributions $F_1, F_2, F_3, \dots, F_N$. There exists a variety of credit products among which collateralized debt obligations (CDO) were are most popular and to some extend famous.

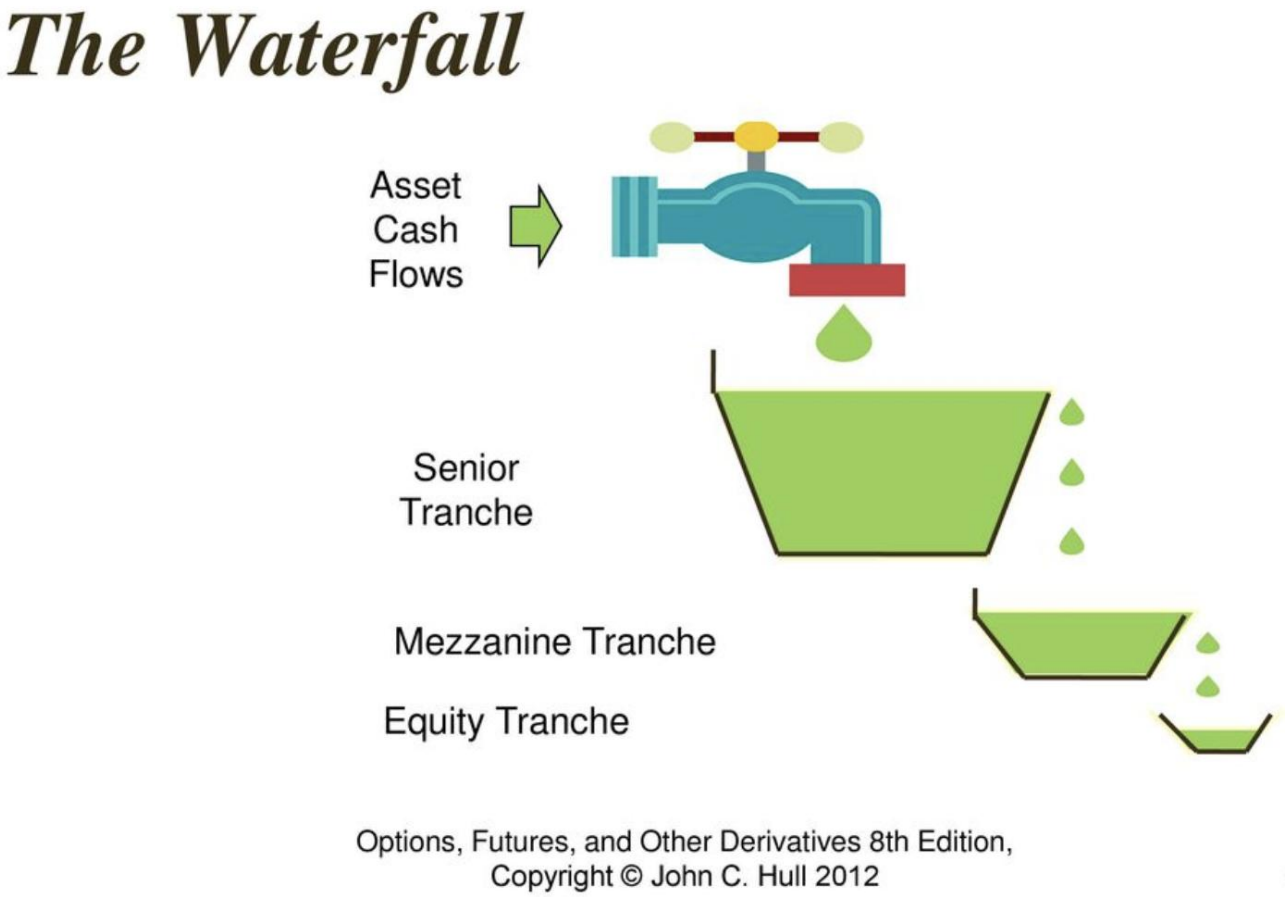
The survival amount associated to the tranche denoted as $[k, K]$ is defined as

$$L_{kK}(t) = \left(KN - LGD \sum_{i=1}^{N} I(\tau_i < t) \right)^+ - \left(kN - LGD \sum_{i=1}^{N} I(\tau_i < t) \right)^+ ,$$

where LGD is the loss-given-default. For CDO tranches we have the following pricing equality

$$Spread \sum_{i=1}^{K} B(0, T_i) EL_{kK}(T_i) = - \int_0^T B(0, t) EL_{kK}(dt) .$$

Collateralized debt obligation



Gaussian copula for credit correlation

If X_1 and X_2 be two Gaussian random variables with mean 0 and variance 1 (normal variable), then there exists a third Gaussian X_0 independent of X_1 such that

$$X_2 = \alpha X_1 + X_0\sqrt{1-\alpha^2}.$$

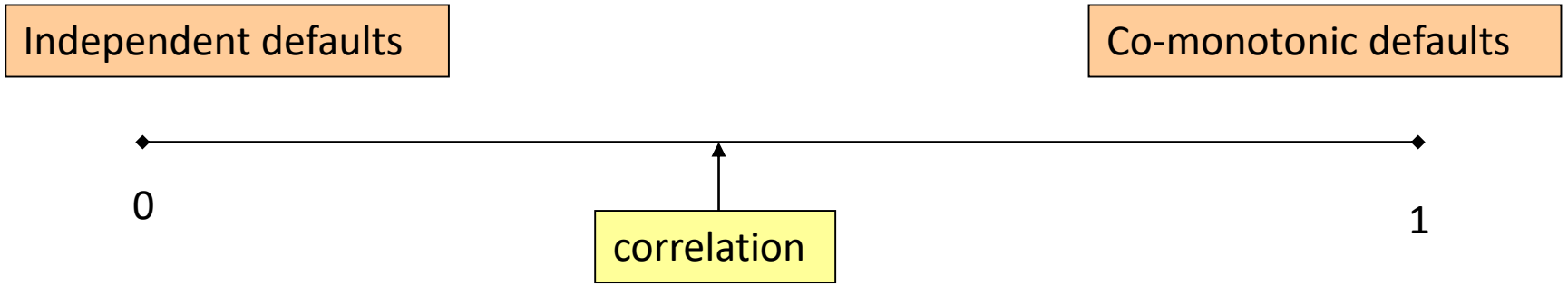
For $\alpha = 1$ we have comonotonicity, for $\alpha = 0$ we have independence. This is a perfect recipe for correlation but the default times $\tau_1, \tau_2, \tau_3, \dots, \tau_N$ are never Gaussian since they are positive. This difficulty can be easily overcome using the copula framework [9].

Namely: Let $X_0, X_1, X_2, \dots, X_d$ be independent random variables with standard normal distribution. Define τ_i as

$$\tau_i = F_i^{-1}\left\{\Phi\left(\alpha X_0 + X_i\sqrt{1-\alpha^2}\right)\right\}.$$

Obviously $P(\tau_i < t) = F_i(t)$, for $\alpha = 1$ we have comonotonicity, for $\alpha = 0$ we have independence exactly as in the Gaussian case.

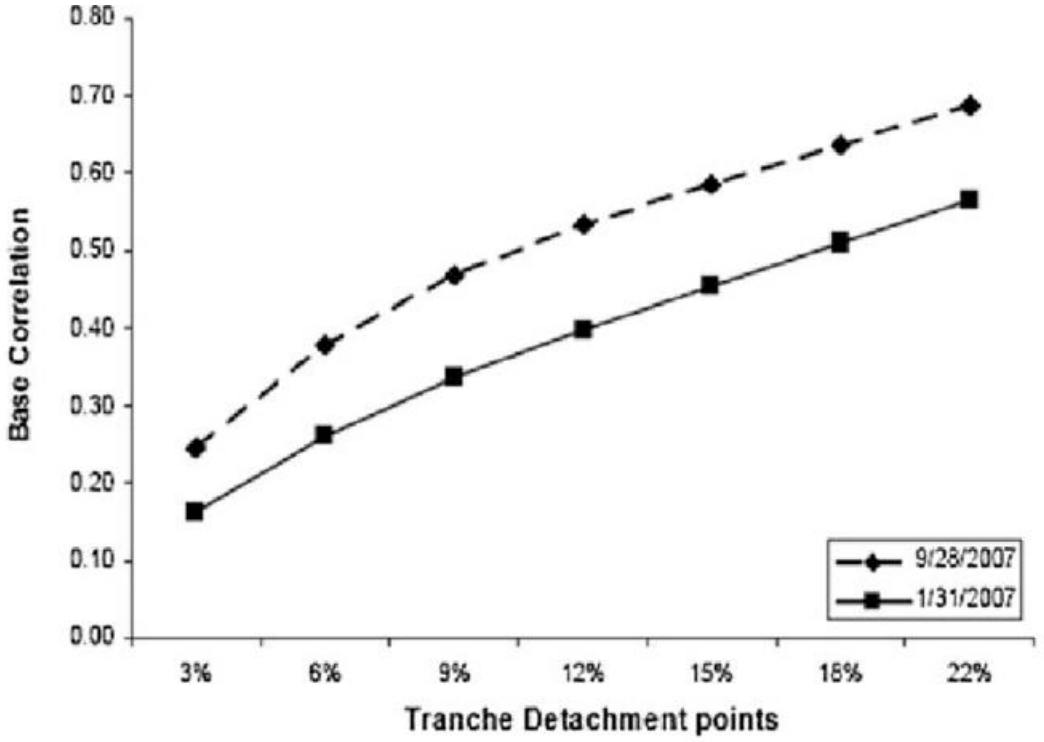
Independent versus co-monotonic



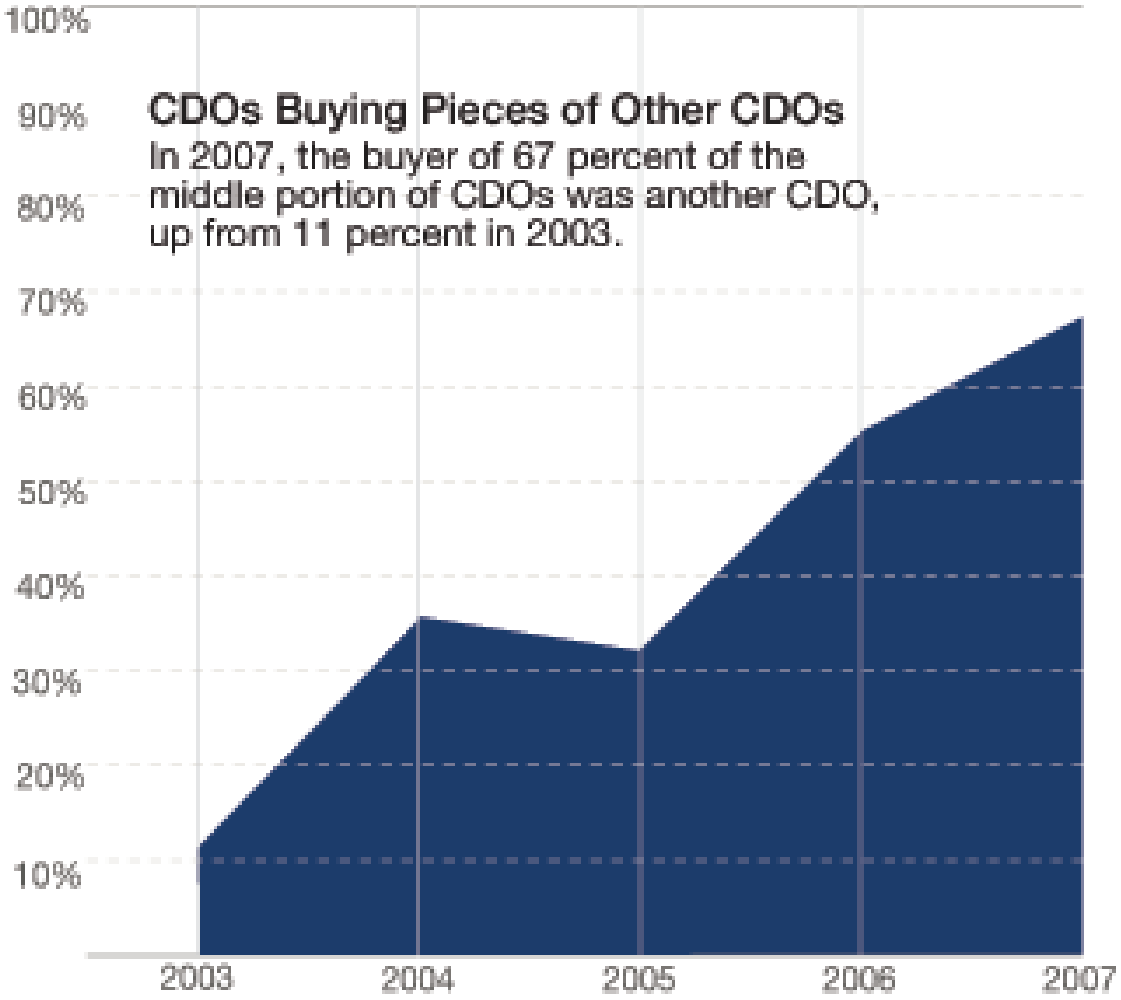
- Independent defaults: All defaults are caused by idiosyncratic factors, for all obligors independent
- Co-monotonic defaults: Defaults are caused by systematic factors driving economy as whole.

Correlation skew

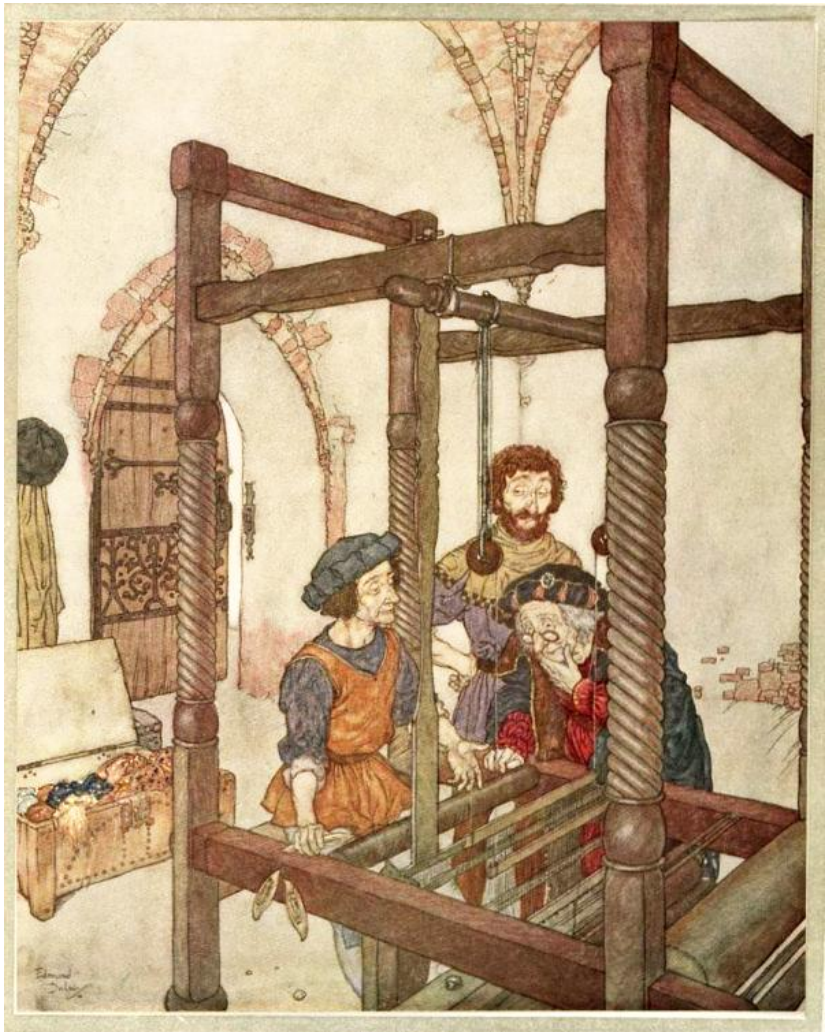
$$Spread(T,k,K)\sum_{i=1}^{K} B(0,T_i)EL_{kK}(\alpha(T,k,K),T_i) = -\int_0^T B(0,t)EL_{kK}(\alpha(T,k,K),dt)\,.$$



CDO squares


CDOs Buying Pieces of Other CDOs
In 2007, the buyer of 67 percent of the middle portion of CDOs was another CDO, up from 11 percent in 2003.
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
2003
2004
2005
2006
2007

Gatheral theory of credit correlation



Big banks and investment firms were not able to correctly price billions-worth securities in their portfolios!

Users versus makers

In contemporary civilization, all devices have two types of users: everyday users, who know how to use the equipment but are unlikely to understand how it works, and designers and maintenance technicians, whose knowledge should be much more in-depth. The same applies to financial models: traders and management only know the rudiments of the models, while knowledge of how they work is reserved for quants. The whole disaster stemmed from the fact that decision makers were convinced that the Gaussian copula was a sufficient tool to model credit risk, when in fact it only supported trading.

Physics envy

In physics we tend to believe that nature is ruled by one principle, so once physicists have one good model, they will not look for another. And even if they do, they will not use both models simultaneously. Social life is different - a model that is good for trading may prove insufficient for portfolio management. And that is exactly what happened.