Principal Zeros of Mittag-Leffler Functions

Their Location and Their Role in the Construction and Analysis of Optimization Algorithms

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Overview

1. Statement of Problem
2. Classical Results on Zeros of Mittag-Leffler Functions
3. Numerical Computations
4. The Case $\alpha \in (1, 2]$
5. Examples
6. Open Questions

Joint work with Komal Usman (THWS)

Basic Definitions

Two-Parameter Mittag-Leffler Functions ($\alpha \geq 1$ and $\beta \geq 0$)

$$E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

(Mittag-Leffler 1903; Wiman 1905)

Properties:

- equal to exp function for $\alpha = \beta = 1$
- entire function of order $1/\alpha$ and type 1
  order of $f = \inf\{m \in \mathbb{R} : f(z) = O(\exp(|z|^m)) \text{ as } |z| \to \infty\}$;
  type of $f = \limsup_{r\to\infty} r^{-(\text{order of } f)} \ln \sup_{|z|\leq r} |f(z)|$
- closely related to eigenfunctions of fractional differential operators

Basic Definitions

Principal Zeros

The number $z_{\alpha,\beta}^{(\mathrm{P})} \in \mathbb{C}$ is called the principal zero of $E_{\alpha,\beta}$ if

$$E_{\alpha,\beta}(z_{\alpha,\beta}^{(\mathrm{P})}) = 0$$

and

$$E_{\alpha,\beta}(z) \neq 0 \quad \text{for all } z \in \mathbb{C} \setminus \{z_{\alpha,\beta}^{(\mathrm{P})}\} \text{ with } |z| \leq |z_{\alpha,\beta}^{(\mathrm{P})}|$$

(i.e. $z_{\alpha,\beta}^{(\mathrm{P})}$ is the unique zero of $E_{\alpha,\beta}$ closest to the origin).

Principal Zeros in Optimization

Assumptions:

- $f : \mathbb{R}^d \to \mathbb{R}$ is strictly convex, differentiable everywhere and μ-gradient dominated with some $\mu > 0$ (i.e. $\|\nabla f(x)\|^2 \geq \mu \left(f(x) - \min_{\xi \in \mathbb{R}^d} f(\xi) \right)$ for all $x \in \mathbb{R}^d$)
- $0 < \alpha < 1$
- $\rho > 0$

Gradient Method with Fractional Learning Rate (Chen et al. 2023)

Find $x : [0, \infty) \to \mathbb{R}^d$ satisfying

$$\dot{x}(t) = -\theta(t) \frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^2}$$

D_0^α = Caputo derivative of order α

where the learning rate θ solves the fractional initial value problem

$$D_0^\alpha \theta(t) = \rho \|\nabla f(x(t))\|^2, \qquad \theta(0) = 0.$$

Fractional Operators

Definition (Riemann-Liouville Integral)

For $\alpha > 0$ and $y \in C[0, b]$,

$$J_0^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} y(\tau) \, \mathrm{d}\tau$$

Definition (Caputo Differential Operator)

For $\alpha > 0$ and $y \in AC^{\lceil \alpha \rceil}[0, b]$,

$$D_0^\alpha y(t) := J_0^{\lceil \alpha \rceil - \alpha} y^{(\lceil \alpha \rceil)}(t).$$

Remark

Caputo differential operators are non-local!

Principal Zeros in Optimization

Theorem (Chen et al. 2023)

Under the given assumptions, the algorithm described above is finite-time convergent, i.e. for any initial guess $x(0)$ there exists some $T \in (0, \infty)$ such that the function x constructed in the algorithm satisfies

$$f(x(t)) = \min_{\xi \in \mathbb{R}^d} f(\xi) \quad \text{for all } t \geq T.$$

More precisely, the method is fixed-time convergent; specifically

$$T \leq \left(\frac{|z^{(\mathrm{P})}_{\alpha+1,1}|}{\rho\mu} \right)^{1/(1+\alpha)}$$

independent of the initial guess $x(0)$.

Principal Zeros in Optimization

Assumptions as above; in addition:

- $0 < \gamma \leq 2$

Gradient Method with Fractional Learning Rate and Generalized Tuning Gain (Chen et al. 2023)

Find $x : [0, \infty) \to \mathbb{R}^d$ satisfying

$$\dot{x}(t) = -\theta(t) \frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^\gamma}$$

where θ solves the initial value problem

$$D_0^\alpha \theta(t) = \rho \|\nabla f(x(t))\|^\gamma, \qquad \theta(0) = 0.$$

($\gamma = 2$ leads to previous method)

Principal Zeros in Optimization

Theorem (Chen et al. 2023)

Under the given assumptions, the generalized algorithm described above is also fixed-time convergent; specifically

$$T \leq \left(\frac{2|z^{(\mathrm{P})}_{\alpha+1,1}|}{\rho\mu\gamma} \right)^{1/(1+\alpha)}.$$

(Reduces to previous result if $\gamma = 2$.)

- parameter μ depends on given objective function
- parameters α, ρ and γ can be chosen by user

Properties of Algorithm

$$\dot{x}(t) = -\theta(t)\frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^{\gamma}},$$
$$D_0^{\alpha}\theta(t) = \rho\|\nabla f(x(t))\|^{\gamma}, \qquad \theta(0) = 0$$

- Memory property of fractional operator (fading memory) helps reduce risk of sudden large overshoots
- Fractional operator acts as low-pass filter, thus smoothes out small high frequency oscillations near optimum
- Memory helps to mitigate effects of noise in gradient

Principal Zeros in Optimization

$$\dot{x}(t) = -\theta(t)\frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^{\gamma}}, \qquad \text{singular when } \nabla f(x(t)) = 0$$

$$D_0^{\alpha}\theta(t) = \rho\|\nabla f(x(t))\|^{\gamma}, \qquad \theta(0) = 0$$

Regularization: Choose $\delta > 0$ and set

$$\dot{x}(t) = -\theta(t)\frac{\nabla f(x(t))}{\max\{\|\nabla f(x(t))\|^{\gamma}, \delta\}},$$

$$D_0^{\alpha}\theta(t) = \rho \max\{\|\nabla f(x(t))\|^{\gamma}, \delta\}, \qquad \theta(0) = 0$$

Theorem (Chen et al. 2023)

Under the assumptions above, the regularized method reaches the bounded region $\left\{x \in \mathbb{R}^d : \mu\left(f(x) - \min_{\xi} f(\xi)\right) \leq \delta^2\right\}$ *in fixed time*

$$T \leq \left(\frac{2|z^{(P)}_{\alpha+1,1}|}{\rho\mu\gamma}\right)^{1/(1+\alpha)}.$$

Principal Zeros in Optimization

Alternative nonsingular method (Powerball-type):

$$\dot{x}(t) = -\theta(t) p(1-\gamma; \nabla f(x(t))),$$
$$D_0^\alpha \theta(t) = \rho \|\nabla f(x(t))\|^\gamma, \qquad \theta(0) = 0$$

where $p(\sigma, y) = (|y_1|^\sigma \operatorname{sgn}(y_1), \ldots, |y_d|^\sigma \operatorname{sgn}(y_d))^\mathsf{T}$
(Same assumptions as above)

Theorem (Chen et al. 2023)

Under the given assumptions, the Powerball-type algorithm is finite-time convergent and fixed-time convergent with

$$T \leq \left(\frac{2|z^{(\mathrm{P})}_{\alpha+1,1}|}{\rho \mu \gamma} \right)^{1/(1+\alpha)}.$$

Principal Zeros and Fractional DEs

Boundary Value Problems for Fractional DEs (Di. 2025)

Let $\alpha \in (1, 2]$, and consider the Caputo-type BVP

$$D^{\alpha} y(t) = f(t, y(t)), \qquad y(0) = b_0, \quad y(T) = b_1,$$

for $t \in [0, T]$ where f is Lipschitz w.r.t. y with Lipschitz constant L.

If

$$LT^{\alpha} < \min\left\{ \left| z^{(\mathrm{P})}_{\alpha,1} \right|, \left| z^{(\mathrm{P})}_{\alpha,\alpha} \right|, \left| z^{(\mathrm{P})}_{\alpha,2} \right| \right\},$$

then

- the BVP has a unique solution, and
- a suitably designed shooting method will converge towards this solution.

Known Results

Theorem

- *If* $\beta = 0$ *then* $z_{\alpha,\beta}^{(\mathrm{P})} = 0$ *is the principal zero of* $E_{\alpha,\beta}$.
- *If* $\alpha \geq 1$ *and* $\beta > 0$ *are such that* $E_{\alpha,\beta}$ *has a principal zero* $z_{\alpha,\beta}^{(\mathrm{P})}$ *then* $z_{\alpha,\beta}^{(\mathrm{P})} \in (-\infty, 0)$.
- *In particular, principal zeros (if they exist) are always real and never positive.*

Key Questions

- For which α and β does $E_{\alpha,\beta}$ have real zeros?
- How many real zeros does $E_{\alpha,\beta}$ have in these cases?
- When does $E_{\alpha,\beta}$ have a principal zero?
- Where is the principal zero $z^{(\mathrm{P})}_{\alpha,\beta}$ located?
- How does $z^{(\mathrm{P})}_{\alpha,\beta}$ depend on α and β?

Goal of this talk: Provide at least partial answers to these questions.

Special Cases

On the interval $(-\infty, 0)$:

- $\alpha = \beta = 1$: $E_{\alpha,\beta}(z) = \exp(z)$—no zeros
- $\alpha = 1$, $\beta \in (0, 1)$: $E_{\alpha,\beta}$ has at least one real zero
- $\alpha = 2$, $\beta = 1$: $E_{\alpha,\beta}(z) = \cos\sqrt{-z}$—infinitely many zeros, precisely known locations
- $\alpha = \beta = 2$: $E_{\alpha,\beta}(z) = \frac{1}{\sqrt{-z}} \sin\sqrt{-z}$—infinitely many zeros, precisely known locations
- $\alpha > 1$, $\beta \in (0, \alpha]$: $E_{\alpha,\beta}$ has at least one real zero
- $\alpha > 2$, $\beta \in (0, 2\alpha - 1]$:
  $E_{\alpha,\beta}$ has infinitely many simple zeros in $(-\infty, 0)$
  (and neither any multiple zeros nor zeros outside of this interval).

  Popov & Sedletskiĭ (2011/2013); see also Pólya (1921) for the case $\alpha \in \mathbb{N}$

- $\alpha \in (1, 2)$, $\beta = 1$: $E_{\alpha,\beta}$ has an odd (finite) number of zeros; precise locations unknown
- $\alpha \in (1, 2)$, $\beta \geq \alpha + 1$: $E_{\alpha,\beta}$ has no real zeros

Information is very limited for the case $\alpha, \beta \in (1, 2]$

Fundamental Idea

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

relevant case: $z \in \mathbb{R}$, $z < 0$

- summands of series expansion alternate in sign
- Leibniz' criterion: if $\left(\left| \frac{z^k}{\Gamma(\alpha k+\beta)} \right| \right)_{k=N}^{\infty}$ is monotonic, then

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \approx \sum_{k=0}^{N} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{and} \quad |\text{error}| < \frac{|z|^{N+1}}{\Gamma(\alpha(N+1) + \beta)}$$

- Monotonicity can be shown for

$$N \geq N_0(\alpha, \beta, z) := 1 + \left\lfloor \frac{1}{\alpha} \max\{1 - \beta, |z|^{1/\alpha}\} \right\rfloor$$

Algorithm

Given: α, β, z and accuracy requirement ε

Procedure:

- Compute $N_0(\alpha, \beta, z)$, set $N \leftarrow N_0(\alpha, \beta, z)$
- WHILE $\dfrac{|z|^{N+1}}{\Gamma(\alpha(N+1)+\beta)} > \dfrac{\varepsilon}{2}$

  set $N \leftarrow N+1$
- set $E \leftarrow 0$
- FOR k FROM N TO 0 STEP -1

  set $E \leftarrow E + \dfrac{z^k}{\Gamma(\alpha k+\beta)}$
- return E

compute with accuracy $\varepsilon/(2N+2)$ e.g. with Python's `mpmath` module

Result is guaranteed to be accurate with absolute error $\leq \varepsilon$

Cases of Specific Interest

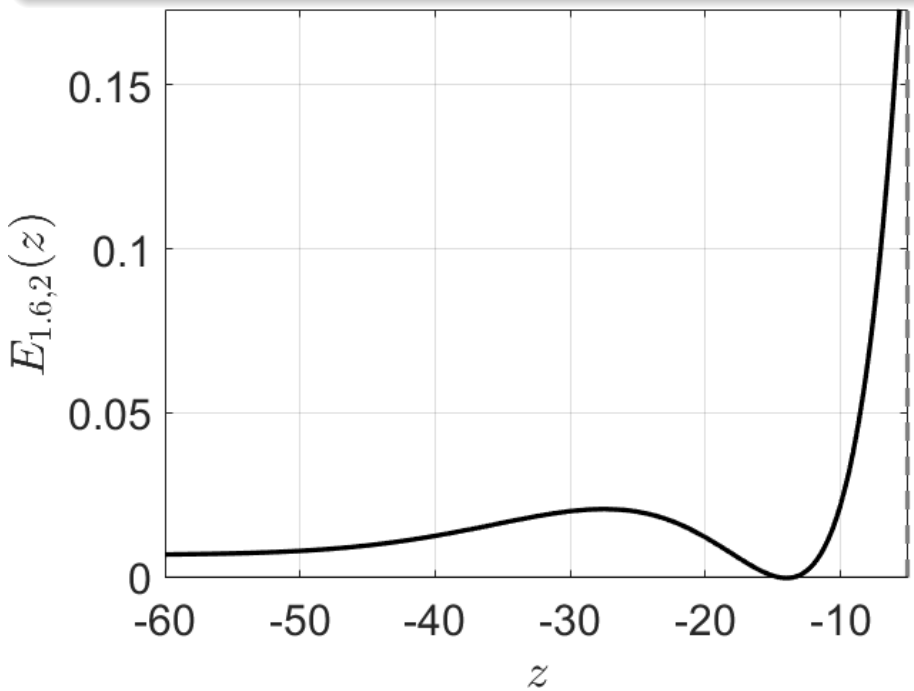
Let $\alpha \in (1, 2]$.
For our example applications above, the following cases are relevant:

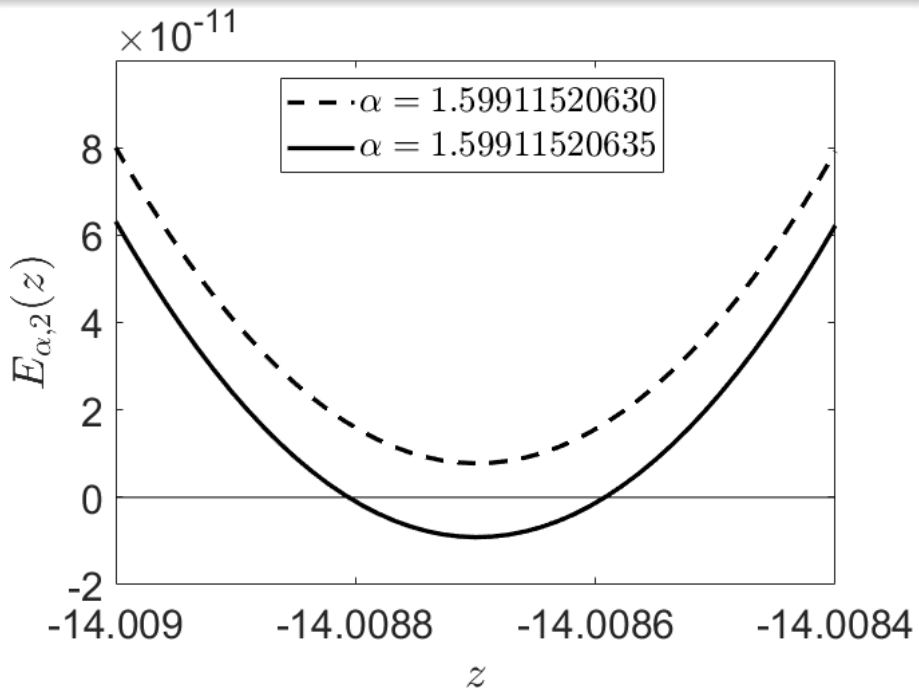
- $\beta = 2$
- $\beta = \alpha$
- $\beta = 1$

$\alpha \in (1, 2]$, $\beta = 2$

Theorem (Popov 2006/2008)

Assume that $\alpha \in (1, 2]$*. Then, there exists some* $\tilde{\alpha} \in (1.55, 1.636)$ *such that* $E_{\alpha,2}$ *possesses a principal zero if and only if* $\alpha \geq \tilde{\alpha}$*.*





Our high precision numerical computations

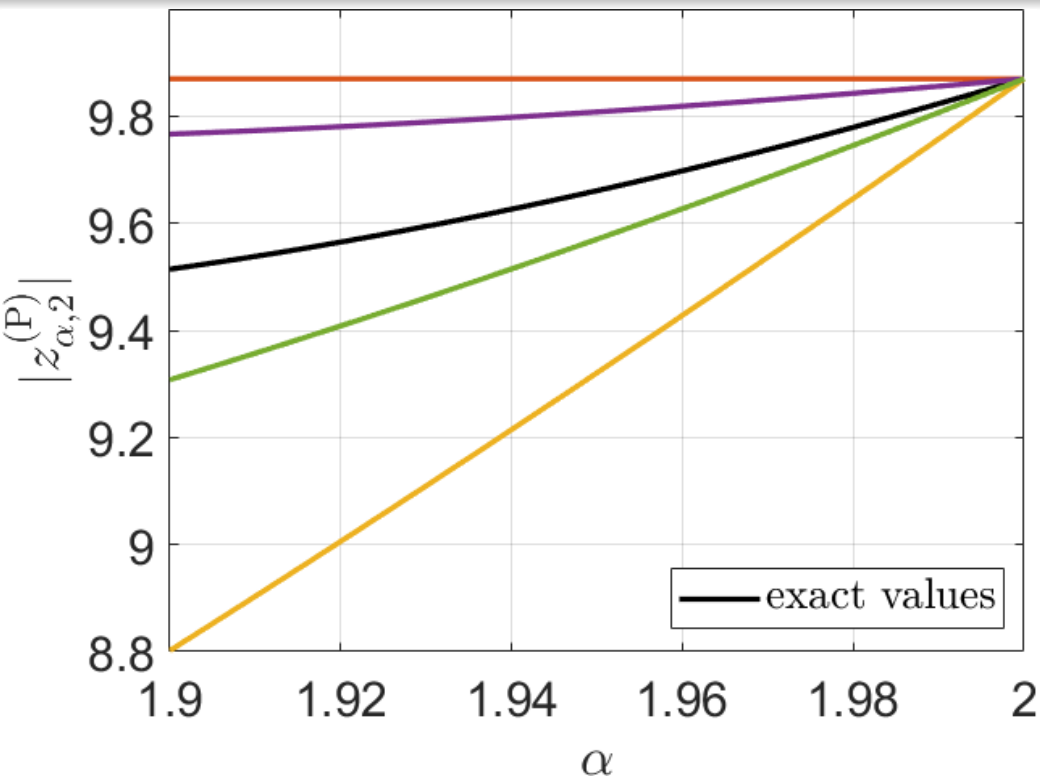
$$\tilde{\alpha} \approx 1.5991152$$

$\alpha \in (1, 2]$, $\beta = 2$

Principal zero of $E_{\alpha,2}$ exists if and only if $\alpha \geq \tilde{\alpha} \approx 1.599$.

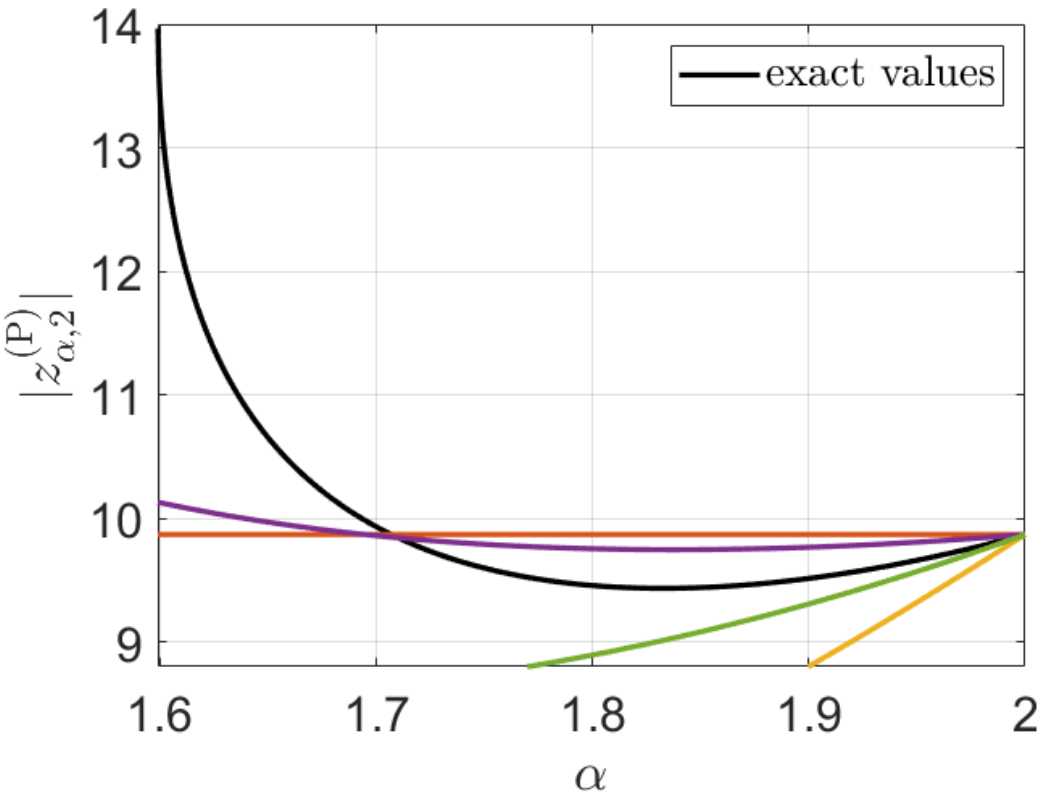
Theorem (Popov 2006/2008)

$$\pi^{\alpha} \leq \left(\frac{\pi \left(\frac{1}{\alpha} + \frac{1}{2} \right)}{\sin \frac{\pi}{\alpha}} \right)^{\alpha} \leq \left| z_{\alpha,2}^{(\mathrm{P})} \right| \leq \left(\frac{2\pi/\alpha}{\sin \frac{\pi}{\alpha}} \right)^{\alpha} \leq \pi^2 \qquad \textit{for all } \alpha \in [1.9, 2].$$



$\alpha \in (1, 2]$, $\beta = 2$

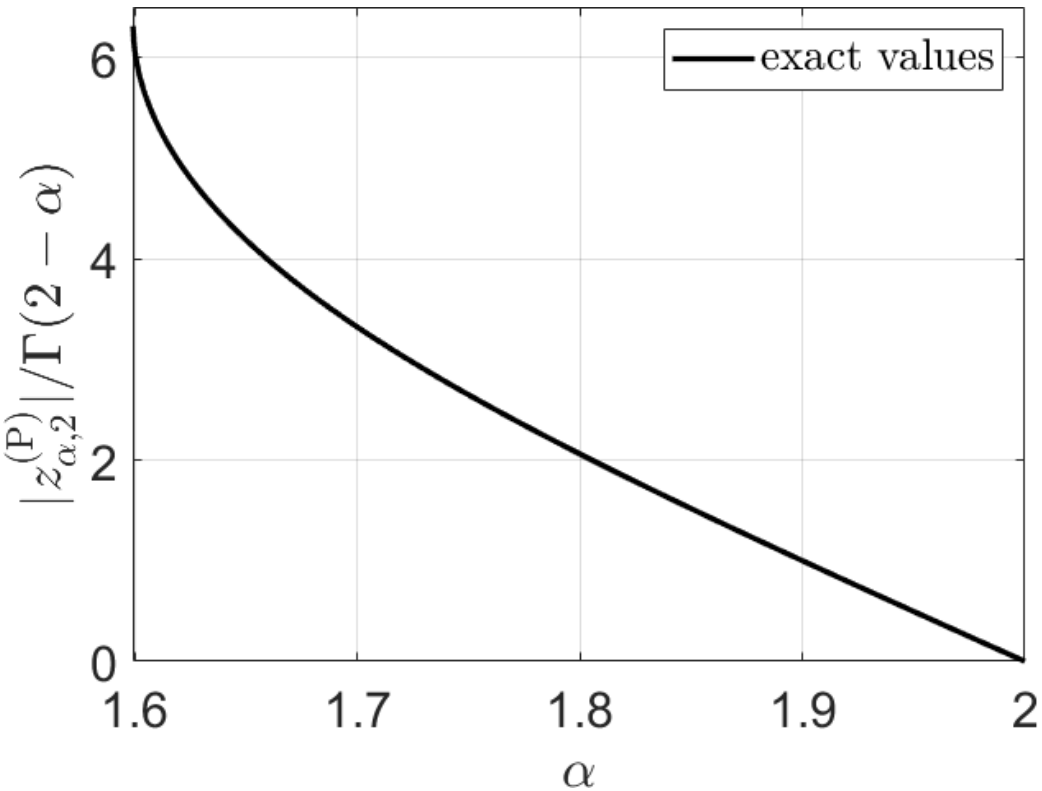
- Principal zero of $E_{\alpha,2}$ exists if and only if $\alpha \geq \tilde{\alpha} \approx 1.599$.
- Popov's Theorem (Slide 25) provides reasonable upper and lower bounds for $z_{\alpha,2}^{(\mathrm{P})}$ when $1.9 \leq \alpha \leq 2$.
- What about the case $\tilde{\alpha} \leq \alpha < 1.9$?



- Popov's lower bounds remain valid but strongly underestimate exact values
- Popov's upper bounds remain valid only for $\alpha \geq 1.71$
- $z_{\alpha,2}^{(\mathrm{P})}$ does not depend on α in a monotonic way
- $z_{\tilde{\alpha},2}^{(\mathrm{P})} \approx -14.0087$

$\alpha \in (\tilde{\alpha}, 2]$, $\beta = 2$

Numerical results:



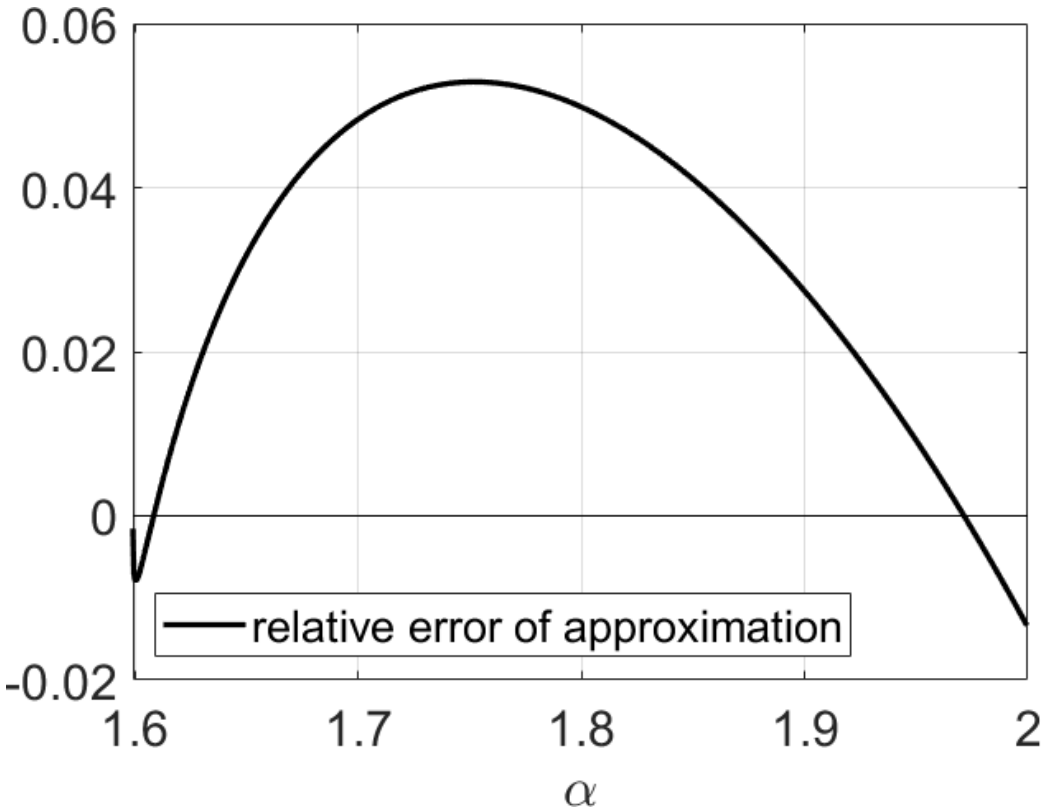
Let $\tilde{z}^{(\mathrm{P})}_{\alpha,2} := \frac{z^{(\mathrm{P})}_{\alpha,2}}{\Gamma(2-\alpha)}$. Then,

- $\tilde{z}^{(\mathrm{P})}_{\alpha,2}$ depends monotonically on α
- $\tilde{z}^{(\mathrm{P})}_{\alpha,2} \to 0$ as $\alpha \to 2$
- Approximation:

$$\tilde{z}^{(\mathrm{P})}_{\alpha,2} \approx \frac{14.0087}{\Gamma(2-\tilde{\alpha})}\left(\frac{1}{2-\tilde{\alpha}}(\alpha-\tilde{\alpha})^{\alpha-1} - 1\right)$$

$\alpha \in (\tilde{\alpha}, 2]$, $\beta = 2$

Numerical results:



Let $\tilde{z}^{(\mathrm{P})}_{\alpha,2} := \frac{z^{(\mathrm{P})}_{\alpha,2}}{\Gamma(2-\alpha)}$. Then,

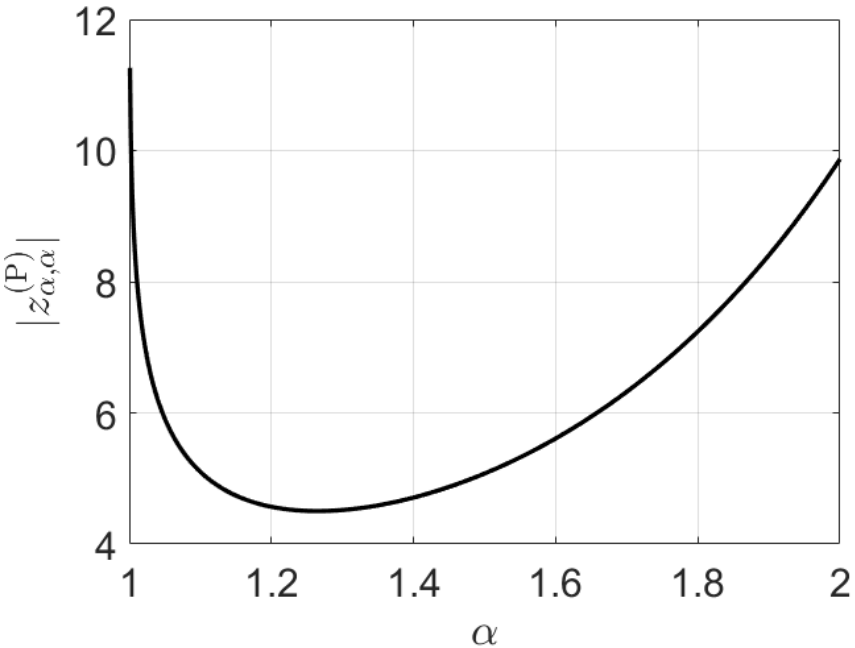
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$$\tilde{z}^{(\mathrm{P})}_{\alpha,2} \approx \frac{14.0087}{\Gamma(2-\tilde{\alpha})}\left(\frac{1}{2-\tilde{\alpha}}(\alpha-\tilde{\alpha})^{\alpha-1} - 1\right)$$

$\alpha \in (1, 2]$, $\beta = \alpha$

Theorem (Hanneken et al. 2013; Eloe & Li 2026)

- $z^{(\mathrm{P})}_{\alpha,\alpha}$ *exists for all* $\alpha \in (1, 2]$.
- $z^{(\mathrm{P})}_{\alpha,\alpha}$ *does not depend on* α *monotonically.*
- $\lim_{\alpha \to 1+} |z^{(\mathrm{P})}_{\alpha,\alpha}| = \infty$.



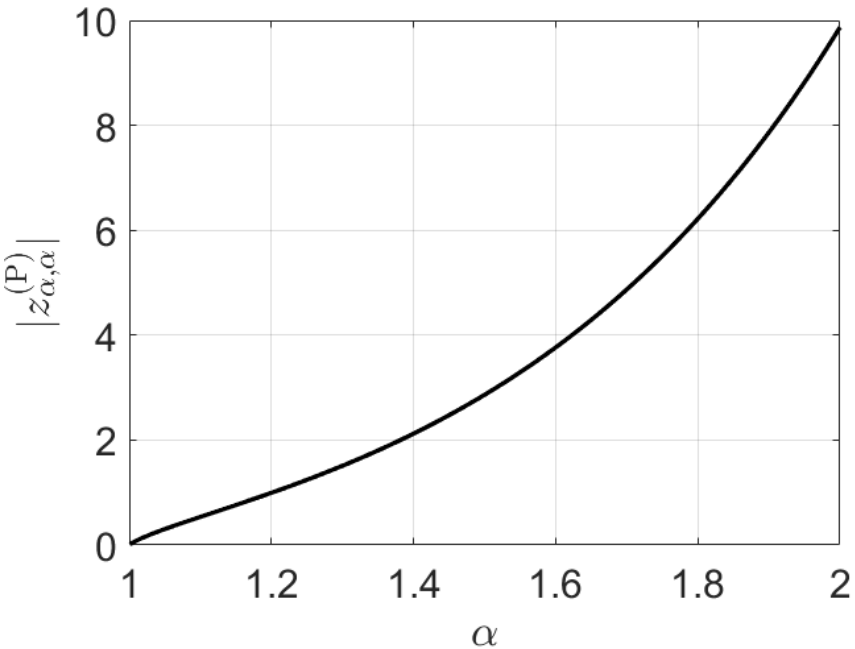
Theorem (Eloe & Li 2026)

Let

$$\tilde{z}^{(\mathrm{P})}_{\alpha,\alpha} := \begin{cases} 0 & \text{for } \alpha = 1, \\ \dfrac{z^{(\mathrm{P})}_{\alpha,\alpha}}{\Gamma(\alpha-1)} & \text{for } \alpha \in (1,2]. \end{cases}$$

Then, the function $\alpha \mapsto \tilde{z}^{(\mathrm{P})}_{\alpha,\alpha}$ *is continuous, bounded and monotonic on* $[1,2]$*. Moreover,*

$$|\tilde{z}^{(\mathrm{P})}_{2,2}| = \pi^2.$$

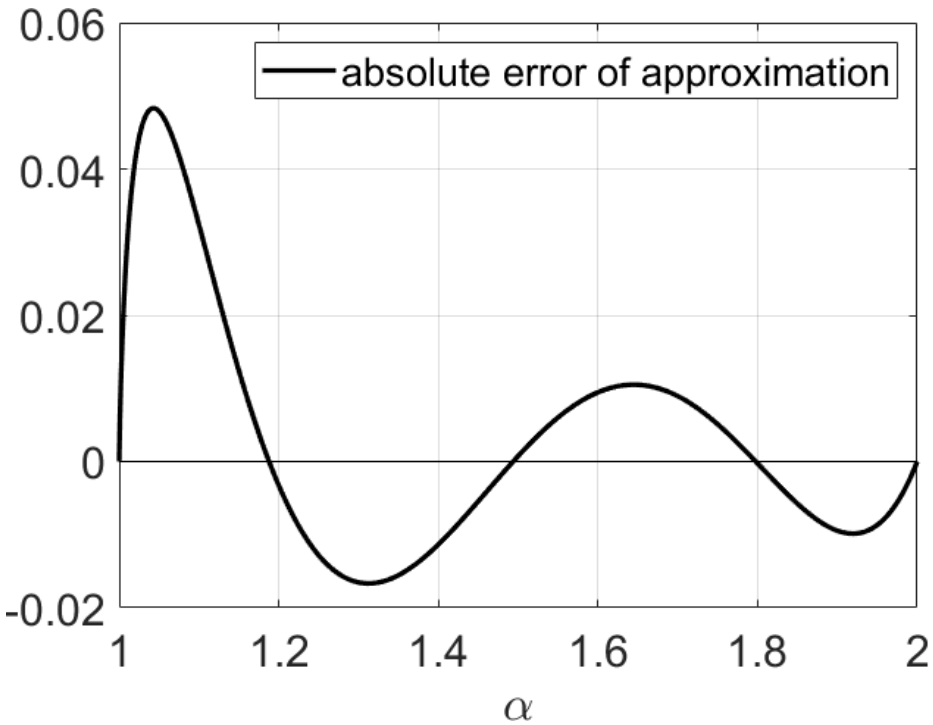


Numerical result

$$|\tilde{z}^{(P)}_{\alpha,\alpha}| \approx k_1(\alpha-1) + (\pi^2 - k_1)(\alpha-1)^2 + k_2(\alpha-1)(\alpha-1.5)(\alpha-2)$$

with $k_1 = 1.58139$, $k_2 = 7.29844$ and

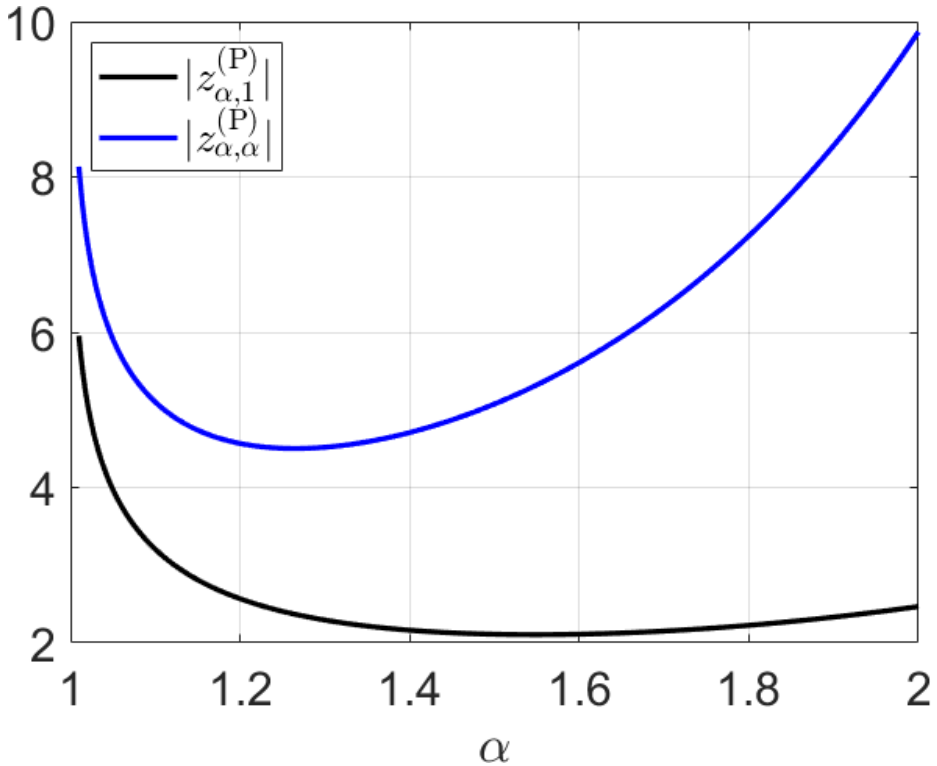
$$|\text{error}| \begin{cases} = 0 & \text{for } \alpha \in \{1,2\}, \\ \leq 0.05 & \text{for all } \alpha \in [1,2]. \end{cases}$$



$\alpha \in (1, 2]$, $\beta = 1$

Theorem (Hanneken et al. 2013; Eloe & Li 2025)

- $z_{\alpha,1}^{(\mathrm{P})}$ *exists for all* $\alpha \in (1, 2]$.
- $z_{\alpha,1}^{(\mathrm{P})}$ *does not depend on* α *monotonically.*
- $z_{2,1}^{(\mathrm{P})} = \pi^2/4$.
- $|z_{\alpha,1}^{(\mathrm{P})}| \leq |z_{\alpha,\alpha}^{(\mathrm{P})}|$ *for all* $\alpha \in (1, 2]$.



Non-monotonicity has consequences for selection of parameters in optimization approaches!

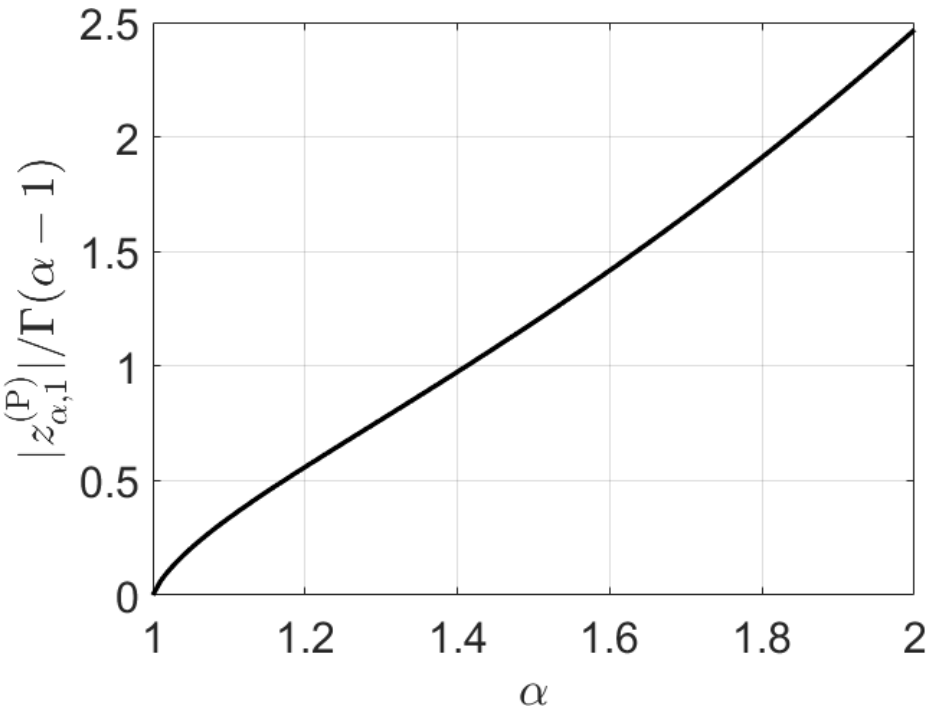
Numerical Results

Let

$$\tilde{z}_{\alpha,1}^{(\mathrm{P})} := \begin{cases} 0 & \text{for } \alpha = 1, \\ \dfrac{z_{\alpha,1}^{(\mathrm{P})}}{\Gamma(\alpha-1)} & \text{for } \alpha \in (1,2]. \end{cases}$$

Then, the function $\alpha \mapsto \tilde{z}_{\alpha,1}^{(\mathrm{P})}$ is continuous, bounded and monotonic on $[1,2]$. Moreover,

$$|\tilde{z}_{2,1}^{(\mathrm{P})}| = \frac{\pi^2}{4}.$$

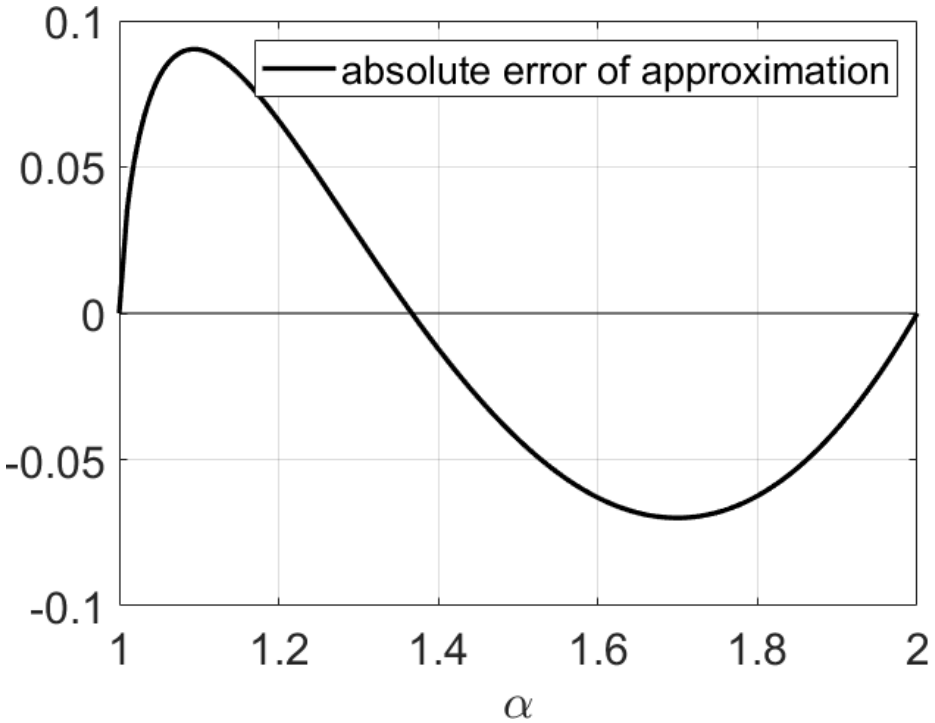


Numerical result

$$|\tilde{z}^{(P)}_{\alpha,1}| \approx \frac{\pi^2}{4}(\alpha - 1)$$

with

$$|\text{error}| \begin{cases} = 0 & \text{for } \alpha \in \{1, 2\}, \\ \leq 0.1 & \text{for all } \alpha \in [1, 2]. \end{cases}$$

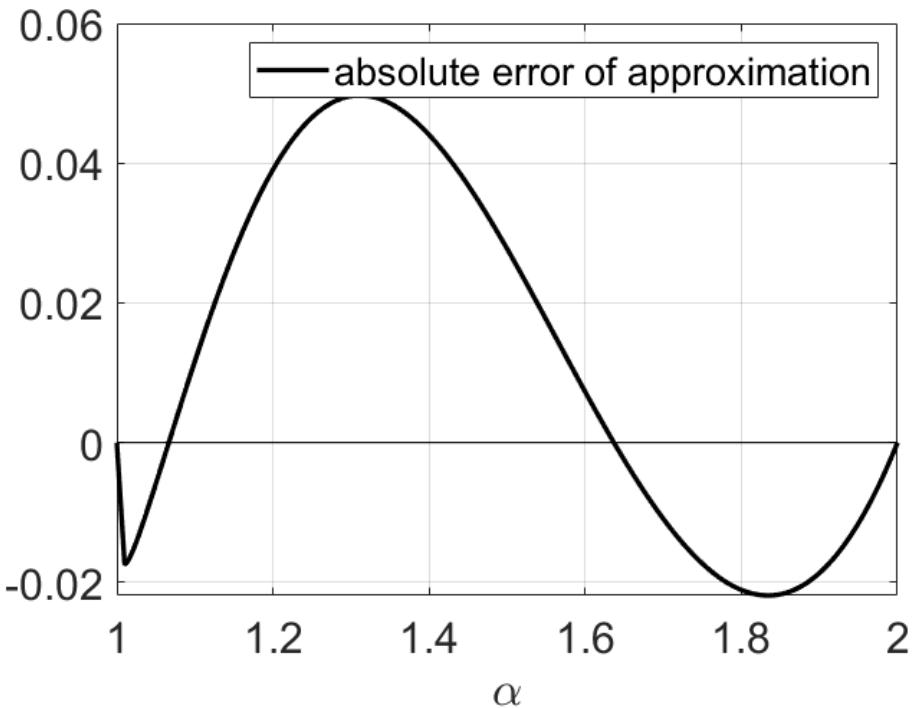


Numerical result

$$|\tilde{z}^{(P)}_{\alpha,1}| \approx \frac{\pi^2}{4}(\alpha - 1) + 0.09\frac{\sin 2\pi\sqrt{\alpha - 1}}{\sqrt{\alpha}}$$

with

$$|\text{error}| \begin{cases} = 0 & \text{for } \alpha \in \{1, 2\}, \\ \leq 0.05 & \text{for all } \alpha \in [1, 2]. \end{cases}$$



Example 1

Find minimum of

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad x \mapsto \exp(x_1 + 2x_2 + 3x_3) + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

Properties of function f:

- strictly convex
- continuously differentiable on complete domain
- μ-gradient dominated with $\mu = 2$

Gradient Method with Fractional Learning Rate and Generalized Tuning Gain

Find $x : [0, \infty) \to \mathbb{R}^d$ satisfying initial value $x(0)$ is arbitrary

$$\dot{x}(t) = -\theta(t) \frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^\gamma}$$

where θ solves the initial value problem

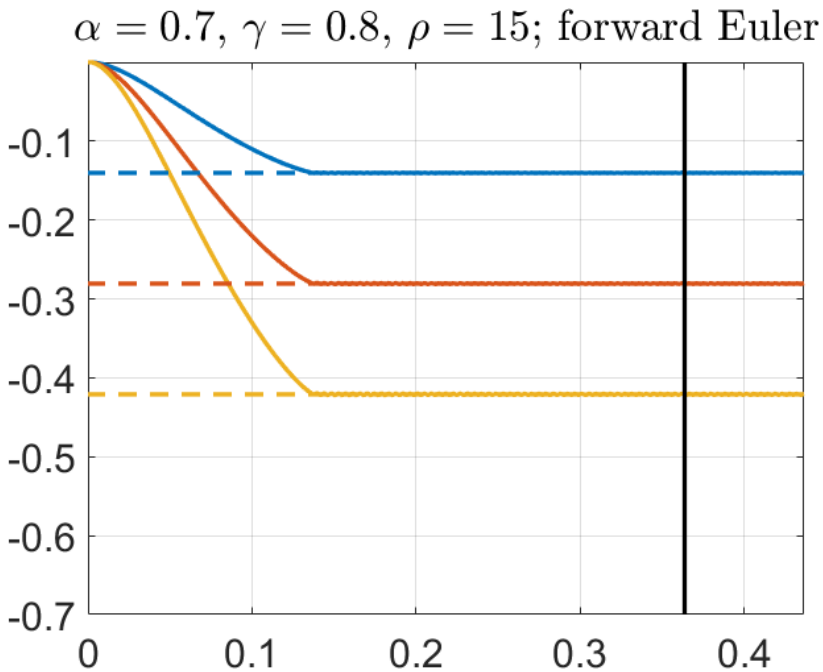
$$D_0^\alpha \theta(t) = \rho \|\nabla f(x(t))\|^\gamma, \qquad \theta(0) = 0.$$

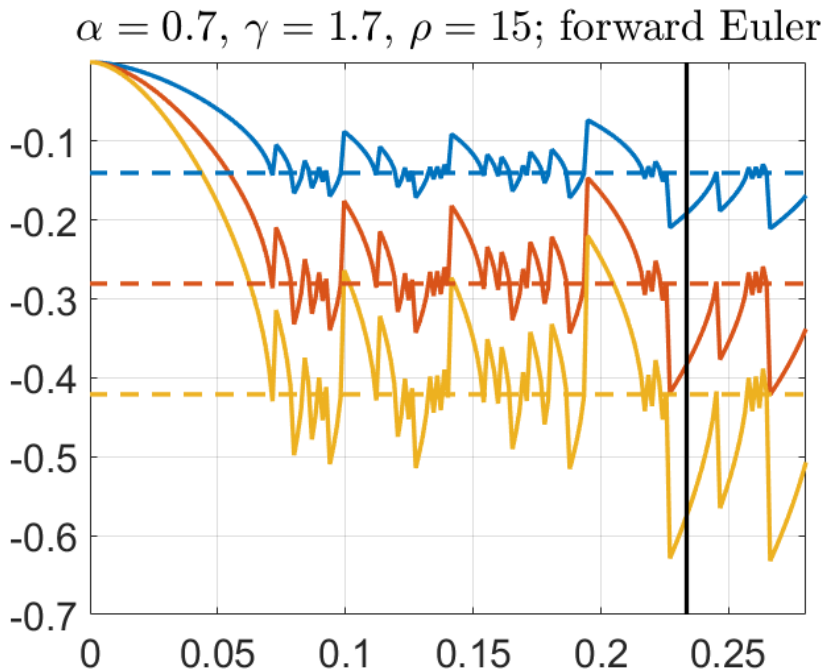
Results for Example 1

$$\dot{x}(t) = -\theta(t)\frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^{\gamma}}, \qquad x(0) = 0,$$
$$D_0^{\alpha}\theta(t) = \rho\|\nabla f(x(t))\|^{\gamma}, \qquad \theta(0) = 0.$$

$(d+1)$-dimensional incommensurate ODE system

Numerical solution:





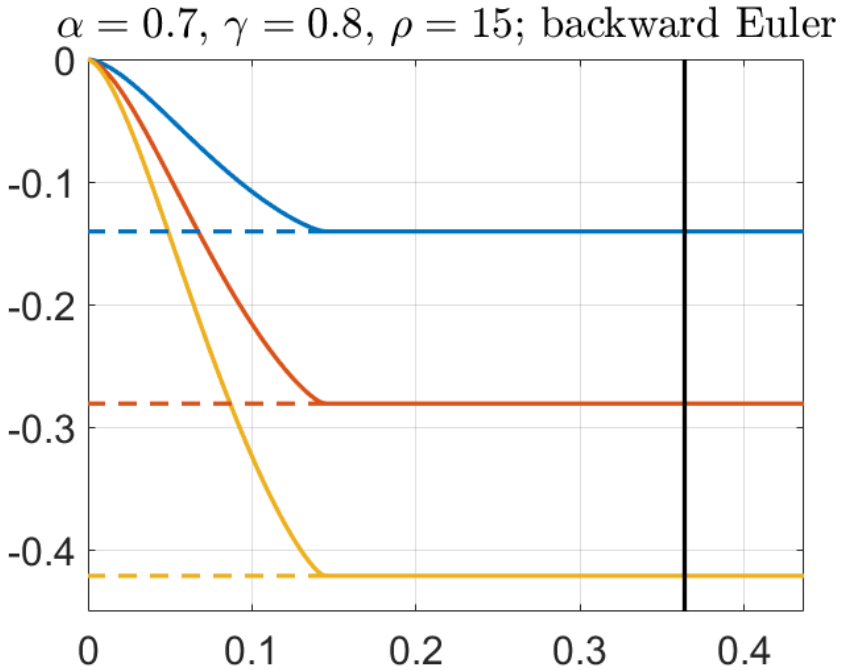
- black vertical line: bound for convergence time
- dashed horizontal lines: coordinates of global minimum

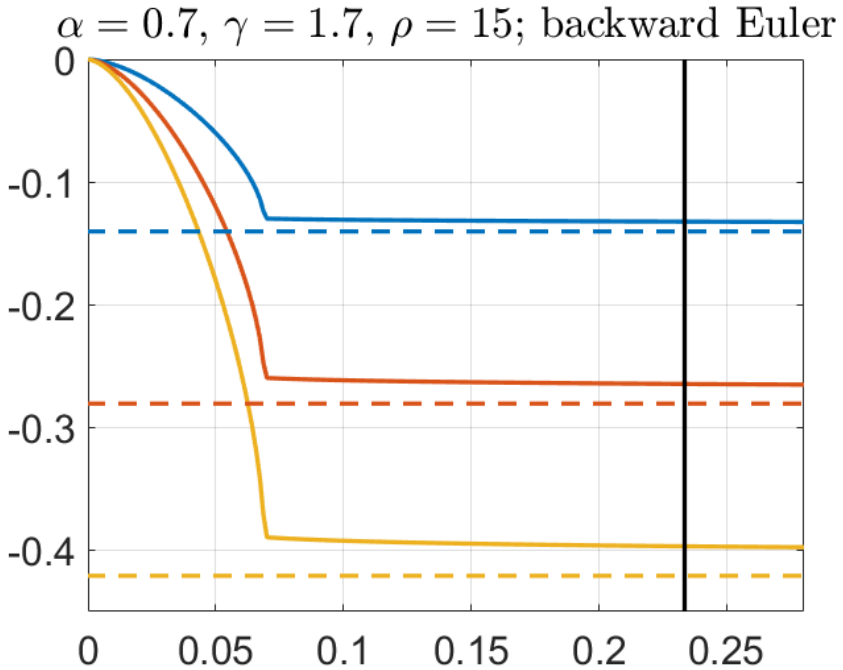
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$$D_0^{\alpha}\theta(t) = \rho\|\nabla f(x(t))\|^{\gamma}, \qquad \theta(0) = 0.$$

$(d+1)$-dimensional incommensurate ODE system

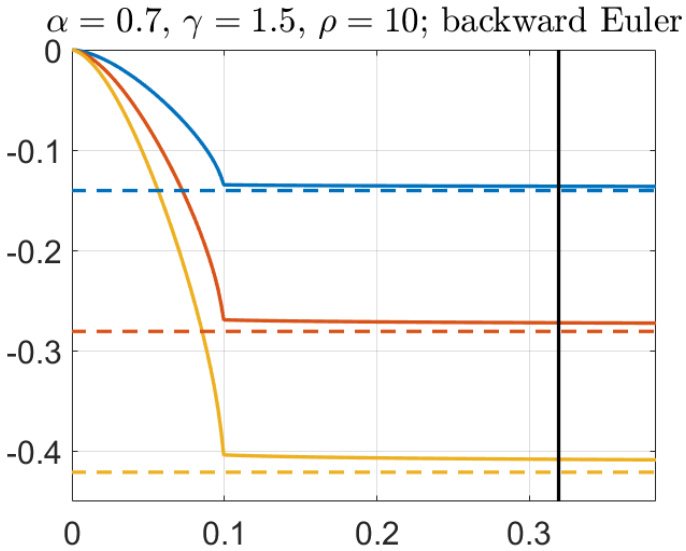
Numerical solution:

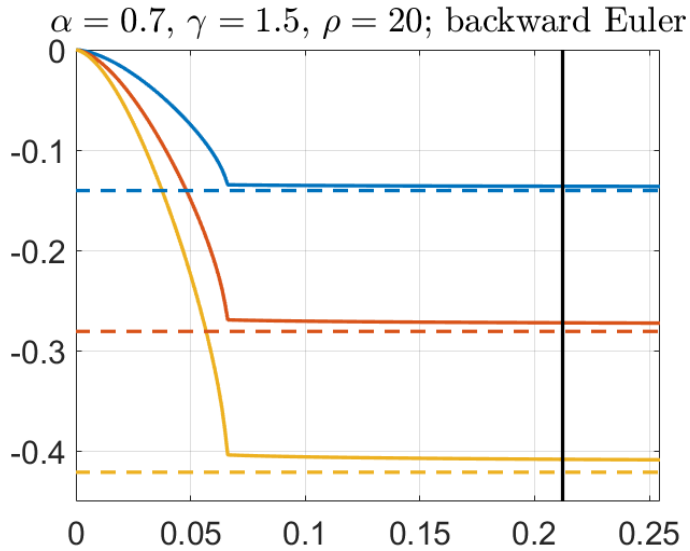


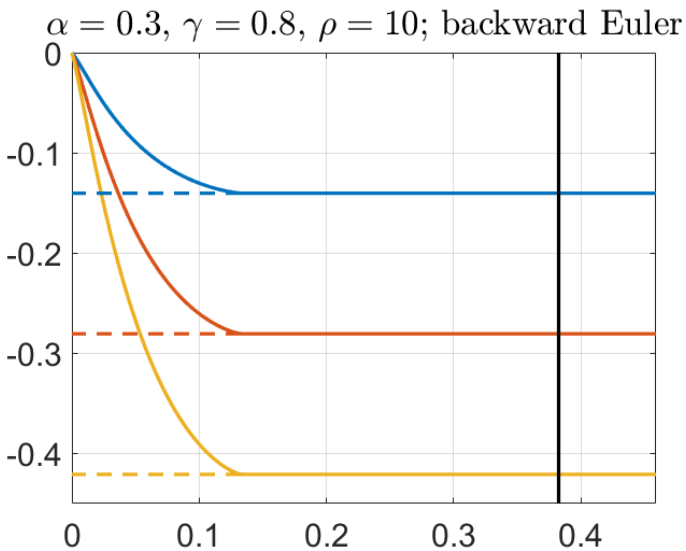


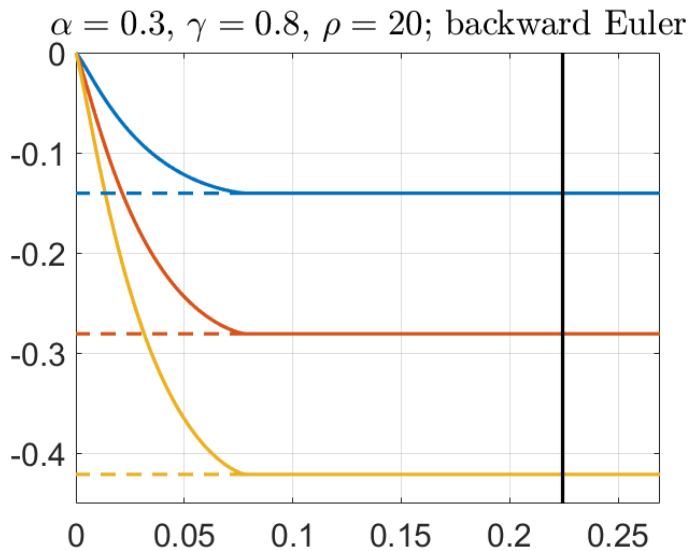
- black vertical line: bound for convergence time
- dashed horizontal lines: coordinates of global minimum

Results for Example 1



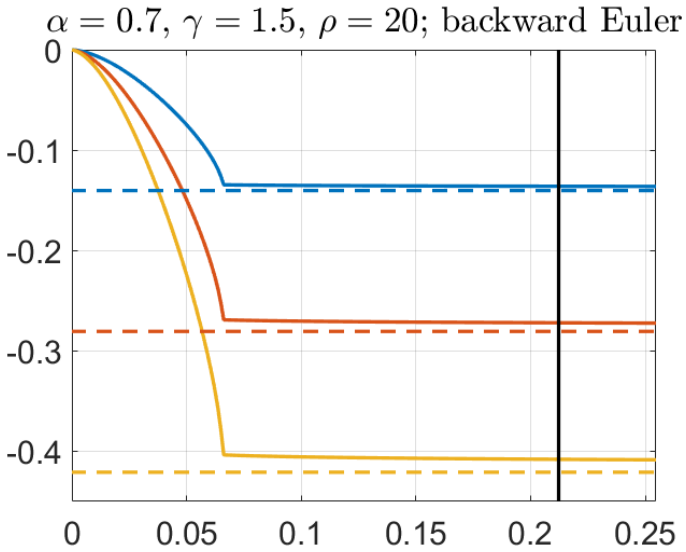


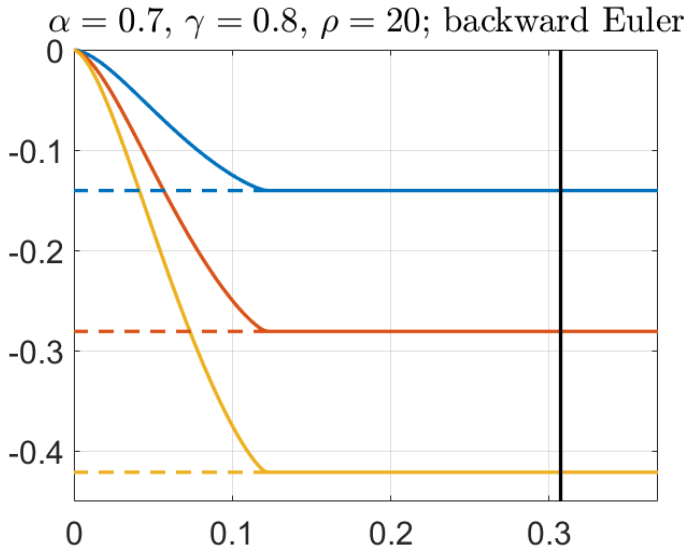


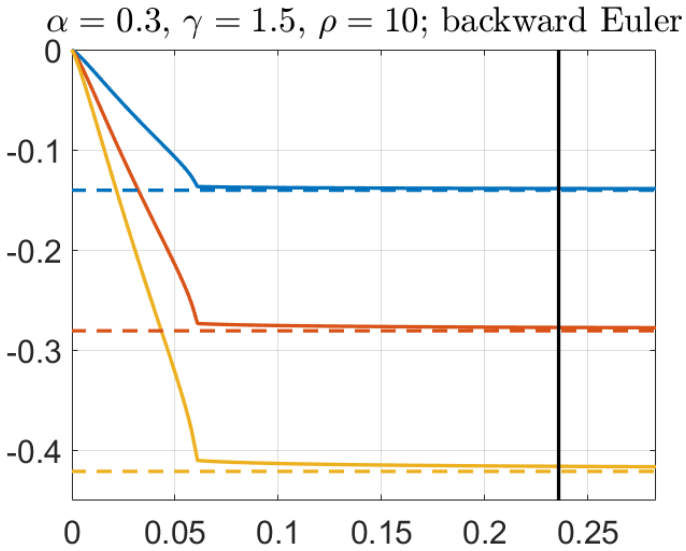


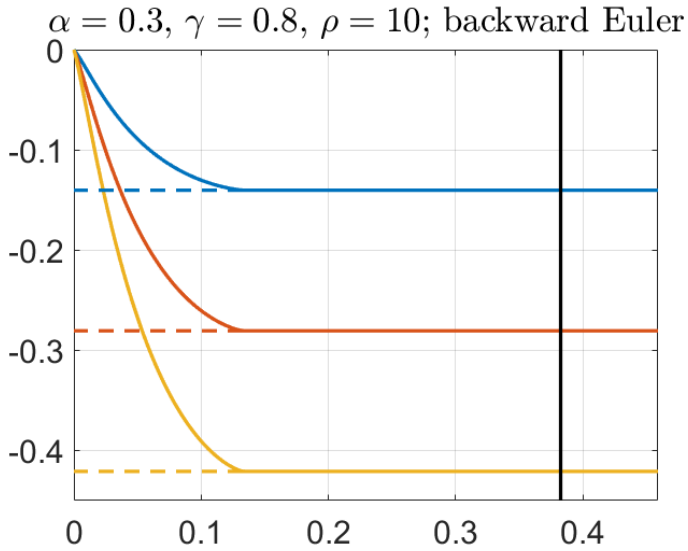
No significant dependence on ρ

Results for Example 1



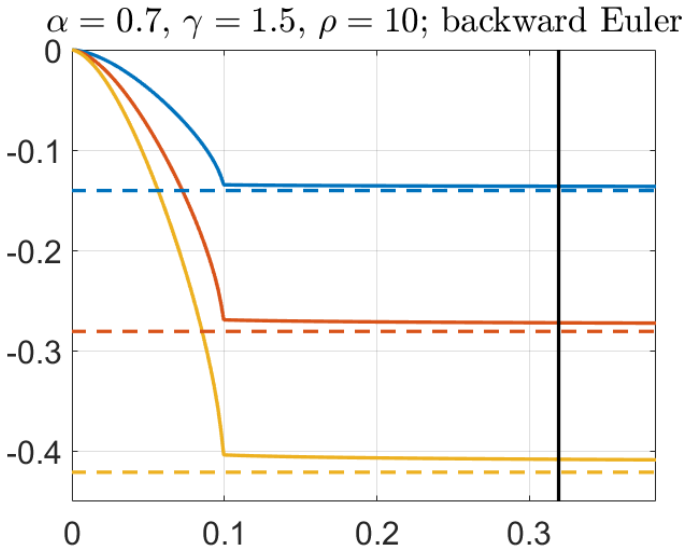


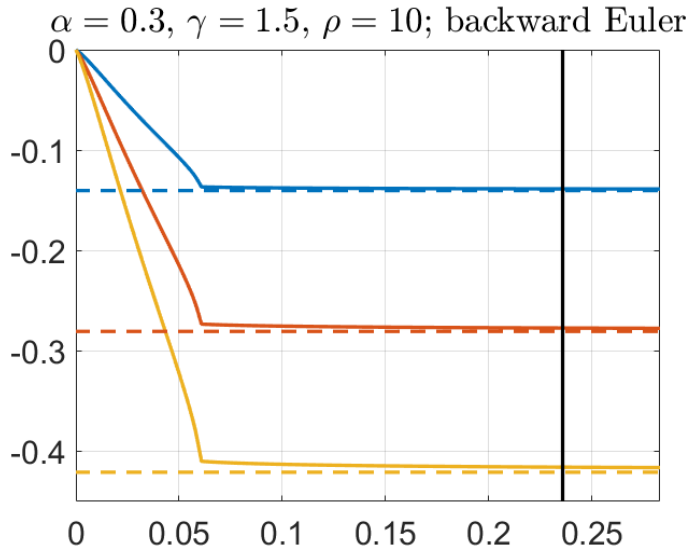


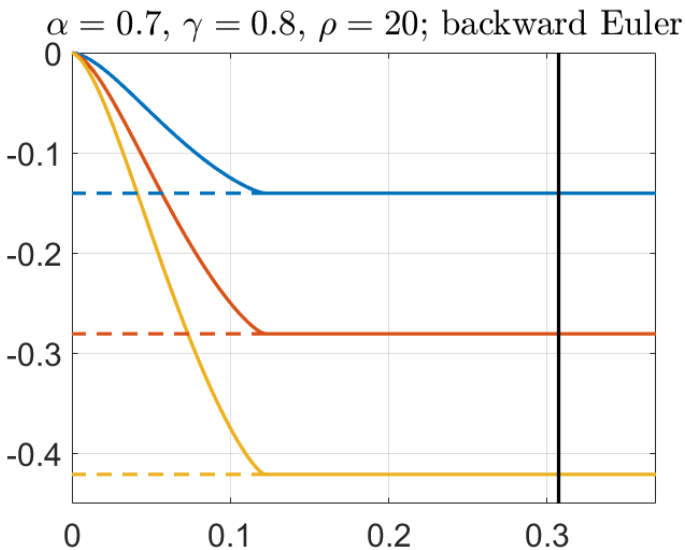


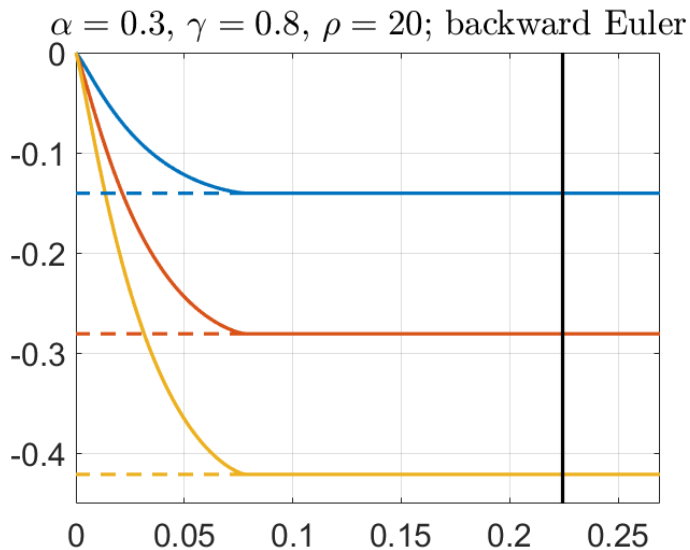
Better convergence when γ is small

Results for Example 1



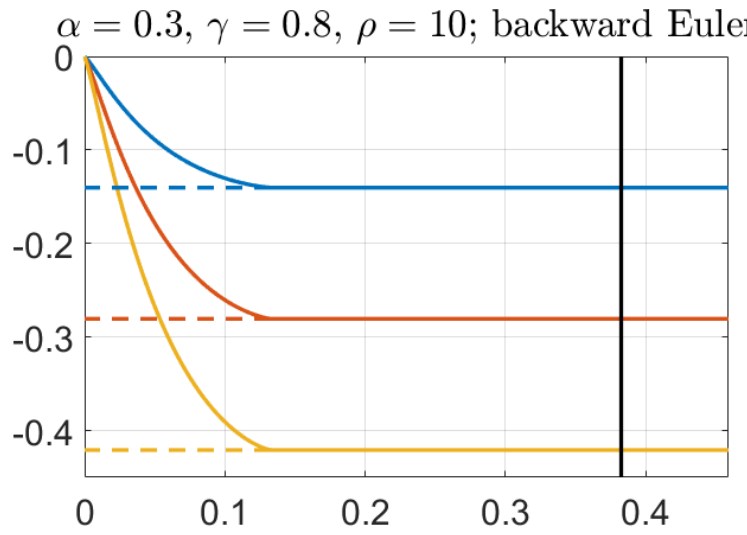


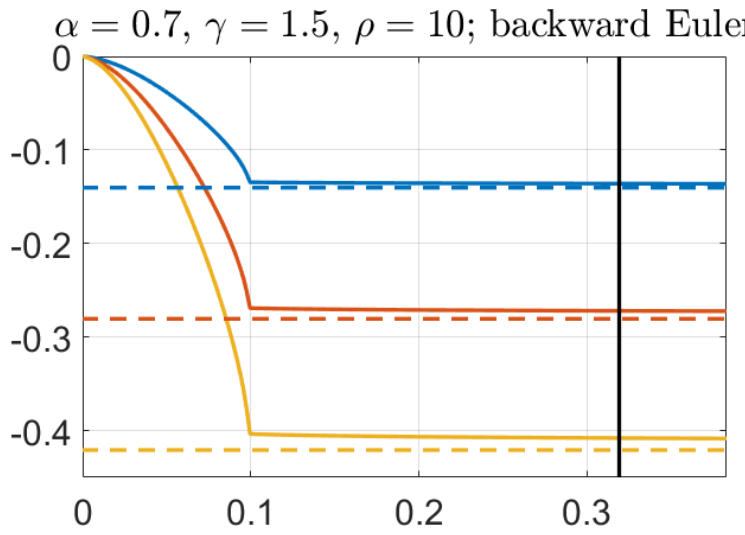


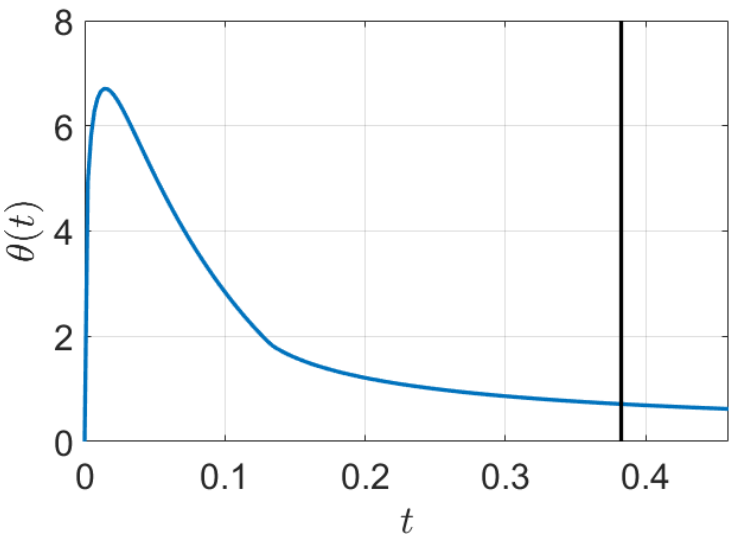


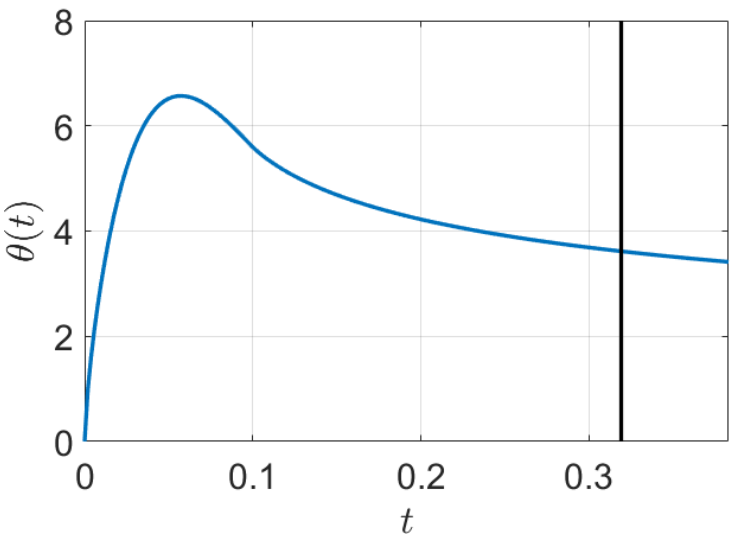
Slightly better convergence when α is small

Results for Example 1









Evolution of learning rate θ

Example 2

Find minimum of

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad x \mapsto (x_1 + 1)^2 + \frac{1}{10}(x_2 + 2)^2 + \frac{x_3^2}{1 + x_3^2}$$

Properties of function f:

- neither convex nor μ-gradient dominated
- possesses unique local minimum
- continuously differentiable on complete domain

Gradient Method with Fractional Learning Rate and Generalized Tuning Gain

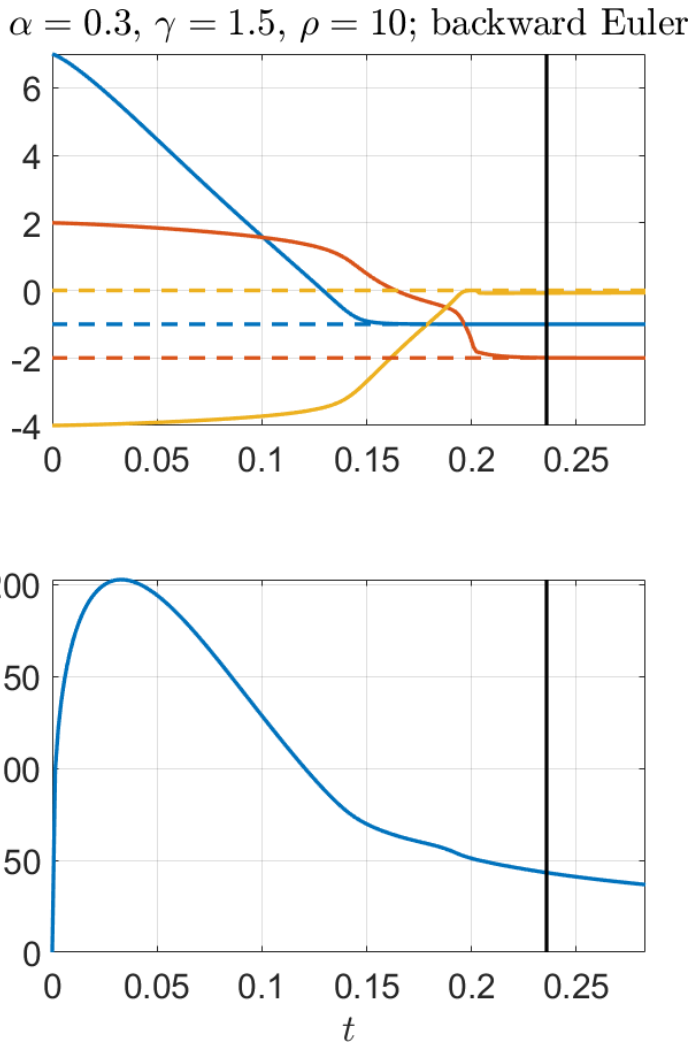
Find $x : [0, \infty) \to \mathbb{R}^d$ satisfying initial value $x(0)$ is arbitrary

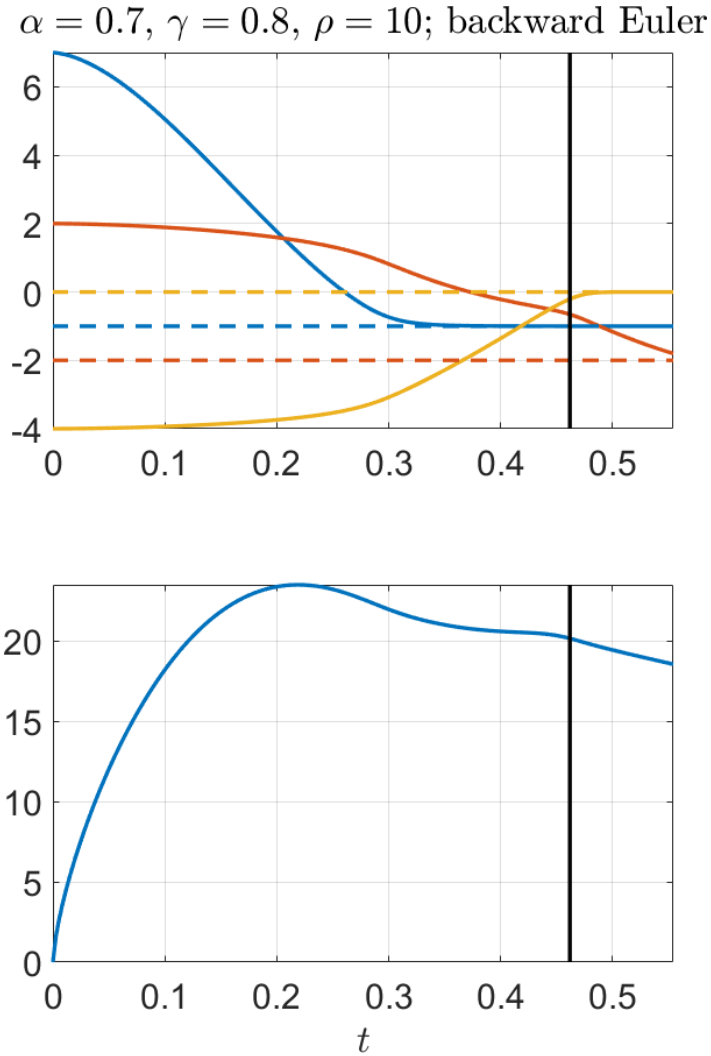
$$\dot{x}(t) = -\theta(t) \frac{\nabla f(x(t))}{\|\nabla f(x(t))\|^\gamma},$$

where θ solves the initial value problem

$$D_0^\alpha \theta(t) = \rho \|\nabla f(x(t))\|^\gamma, \qquad \theta(0) = 0.$$

Results for Example 2





- Changes in ρ and α have only small effects
- Changes in γ have larger effects

Open Questions

- Quantitative and qualitative results on location of zeros of $E_{\alpha,\beta}$ for other values of β
- Proofs for conjectures on monotonicity etc. of principal zeros of Mittag-Leffler functions
- Thorough investigation of convergence behaviour of optimization methods and how it depends on the parameters
- Dependence of suitable stepsize for backward Euler method on the parameters
- Consequences of replacing backward Euler method by other ODE solver

Thank you for your attention!

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kai.diethelm@thws.de

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