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# Mixed-Integer Optimal Control: From ODEs and PDEs to Graphon Dynamical Networks

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# Overview

- Mixed-Integer Optimal Control Problems
- Decomposition, Approximation, Reconstruction
- The ODE control case
- The elliptic PDE control case
- The case of controls on graphon dynamical networks

# MIOCP: A Simplified Setting

Consider a simplified setting (ODE):

$$(MIOCP) \quad \left\{ \begin{array}{l} \min_{y, v} J(y(t)) \\ \text{s.t.} \quad E(y) = f(y(t), v(t)) \\ \quad \quad 0 \leq c(y(t), v(t)) \text{ a.e. } t \in \mathcal{T} \\ \quad \quad v(t) \in \{v^1, \dots, v^M\} \text{ a.e. } t \in \mathcal{T} \end{array} \right.$$

Point of interest: (MIOCP) has a **distributed mesh-dependent discrete-valued** variable  $v$ .

Example:

$$E(y) = \dot{y}, \quad y(0) = y_0,$$

and  $f$  is Lipschitz continuous for all choices of  $v : \mathcal{T} \rightarrow \{v^1, \dots, v^M\}$ .

# MIPDECO: A Simplified Setting

Consider a simplified setting (PDE):

$$\text{(MIPDECO)} \quad \left\{ \begin{array}{l} \min_{y,v} J(y(x)) \\ \text{s.t.} \quad E(y) = f(y(x), v(x)) \\ \quad \quad 0 \leq c(y(x), v(x)) \text{ a.e. } x \in \Omega \\ \quad \quad v(x) \in \{v^1, \dots, v^M\} \text{ a.e. } x \in \Omega \end{array} \right.$$

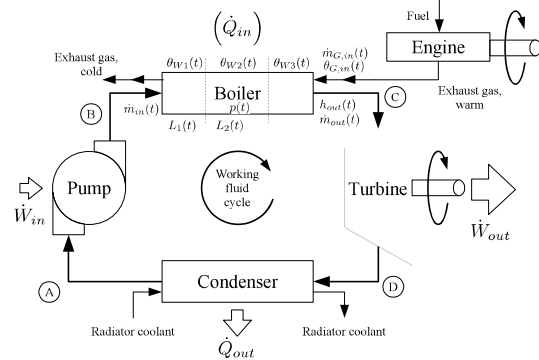
Point of interest: (MIPDECO) has a **distributed (mesh-dependent) discrete-valued** variable  $v$ .

Example:

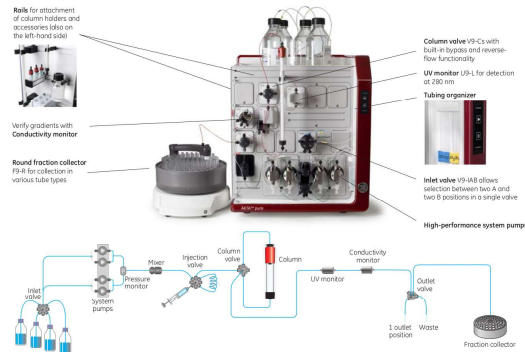
$$E(y) = \dot{y} - Ay, \quad y(0) = y_0,$$

where  $A$  generates a  $C_0$ -semigroup on  $\mathcal{T} = [0, T]$  and  $f$  is Lipschitz continuous for all choices of  $v : \Omega \rightarrow \{v^1, \dots, v^M\}$ .

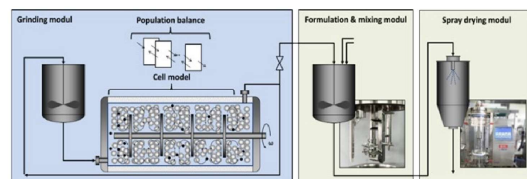
# Ubiquitous in Modeling and Control



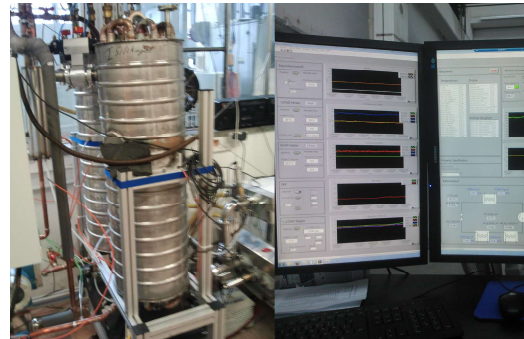
Daimler Truck



Octapharma



IPAT



TLK Thermo



IAV



KuKa Medical Robotics



BASF

... but large-scale,  
nonlinear,  
non-convex, and  
NP-hard.

# Variety of Solution Approaches

- ODE: Direct transcription and **MINLP branching rules** for time dependencies
- PDE: **Inverting the control-to-state law** eliminates  $y$  and results in a MINLP purely in the control domain
- ODE, PDE with boundary controls: **Mode insertion gradients**
- ODE: **Switching Time Optimization** if the order in time of the choices  $v^i$ ,  $1 \leq i \leq M$  is known (optimally, or fixed a priori),
- ODE: **Variable Time Transformation** yields a (continuous) optimal control problem
- ODE, PDE: **Convexification and Relaxation** in the control domain yields a (continuous) optimal control problem

# Direct Transcription to a MINLP

To communicate with solvers, (MIOCP) is discretized on  $N$  elements in time. In the simplest case, one obtains a MINLP with an  $N$ -fold iterated structure:

$$(\text{MINLP}_N) \quad \left\{ \begin{array}{ll} \min_{y,v} J(y_N) \\ \text{s.t.} & y_{n+1} = y_n + \frac{T}{N} \cdot f(y_n, v_n) \quad 0 \leq n \leq N-1 \\ & 0 \leq c(y_n, v_n) \quad 0 \leq n \leq N-1 \\ & 0 \leq c(y_N) \\ & v_n \in \{v^1, \dots, v^M\} \quad 0 \leq n \leq N-1 \end{array} \right.$$

We have  $y \in \mathbb{R}^{n_x \cdot N}$ ,  $v \in \{v^1, \dots, v^M\}^N$ . You get to choose  $N \gg 1$ . One often is interested in the behavior as  $N \rightarrow \infty$ .

# MINLP Branching on Time Dependencies

(Hahn, Leyffer, Zavala, 2017)

Branch-and-bound facilities tailored to the iterated MINLP structure (“arrow of time”):

- **Node selection:** Best-then-dive strategy picks nodes looking at parent’s lower bound. If “best” nodes are found at shallow depth, dives explore huge suboptimal tree branches (number  $N \cdot M$  of integer variables is high).

Better in MIOCP: Depth-first but disregard nodes above a dynamic cutoff threshold finds good initial decisions quickly. Then pick up deferred nodes with best lower bounds.

- **Branching:** Reliability branching with priorities corresponding to time interval numbers. This gives backwards in time branching behavior, which tends to work best.

# State Elimination

(Buchheim, Meyer et al., 2018; Thünen, Leyffer et al., 2022)

For a MIPDECO

$$\begin{aligned} \min_{y,u} \quad & c^T u \\ \text{s.t.} \quad & y = S(u), \quad u \in \mathcal{U} \\ & y(x) \geq y_{\min}(x) \text{ a.e. on } \Omega \end{aligned}$$

the state  $y$  can be eliminated from the problem if **point-wise concavity of the solution operator  $S$**  can be shown.

One such example is

$$Ay + g(y) = \sum_{i=1}^M v^i \psi_i, \text{ on } \Omega$$

with  $g$  pointwise non-decreasing and convex. Extensions to parabolic PDEs are possible.

Then, convex MINLP methods such as Outer Approximation can be applied with great efficiency.

# Mode Insertion Gradient

(Axelsson et al., 2008; Hante et al., 2021)

Given a reference control  $v^{(0)}$ , define the **needle variation**

$$v_{\lambda, \tau, \hat{v}}(t) := \begin{cases} v^{(0)}(t) & \text{if } t \notin [\tau - \frac{\lambda}{2}, \tau + \frac{\lambda}{2}] \\ \hat{v} & \text{if } t \in [\tau - \frac{\lambda}{2}, \tau + \frac{\lambda}{2}]. \end{cases}$$

Use the sensitivity

$$\frac{dJ}{d\lambda^+}(\tau, v) = \lim_{\lambda \searrow 0} \frac{1}{\lambda} (J(v_{\lambda, \tau, \hat{v}}) - J(v))$$

to identify variations that yield descent w.r.t. the objective  $J$ .

$\hat{v}$ : alternate mode to insert

$\tau, \lambda$ : tentative instant and duration of application

# Switching Time Optimization

If the order of appearance of mode in the optimal solution is a priori encoded on a grid  $\{t_i \mid 0 \leq i \leq N\}$  by way of a mapping  $\pi : \{1, \dots, N\} \rightarrow \{1, \dots, M\}$ ,

$$v(t) = v^{\pi(i)} \text{ for all } t \in [t_i, t_{i+1}], \quad 0 \leq i < N,$$

then (MIOCP) becomes a **multi-stage OCP**.

The timing of  $N$  mode switches can be optimized on  $\tau \in [0, 1]$ :

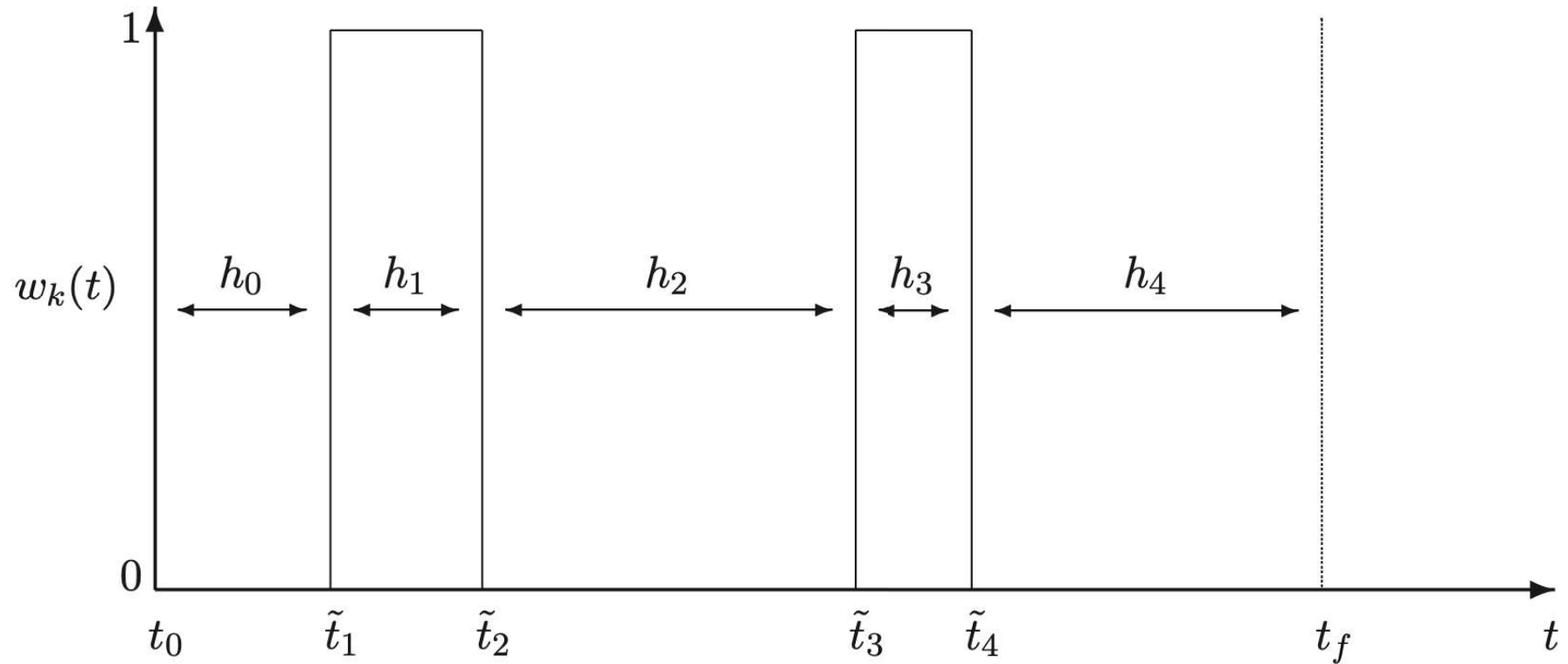
$$\dot{y}(t(\tau)) = h_i \cdot f(t(\tau), y(t(\tau)), v^{\pi(i)}), \quad \tau \in \left[ \frac{i}{N}, \frac{i+1}{N} \right], \quad 0 \leq i < N$$

with the time transformation

$$t(\tau) := \sum_{j=0}^{i-1} h_j + h_i \cdot N \cdot \left( \tau - \frac{i}{N} \right), \quad i := \lfloor N\tau \rfloor.$$

The stage lengths  $h_i \geq 0$  become degrees of freedom.

# Switching Time Optimization



# Variable Time Transformation

(Dubovitskii, Milyutin, 1963; Tartar, 1979; Gerdts et al., 2012)

On a (here equidistant) major grid

$$\{t_i = t_0 + ih \mid 0 \leq i \leq N\}$$

introduce a **minor grid** with  $M$  intervals per major interval,

$$\left\{ \tau_{i,j} = t_i + j \frac{h}{M} \mid 1 \leq j \leq M \right\} \subset [t_i, t_{i+1}], \quad 0 \leq i < N.$$

Let

$$\hat{v}(\tau) = v^j \text{ for } \tau \in [\tau_{i,j-1}, \tau_{i,j}], \quad 0 \leq i < N, 1 \leq j \leq M$$

and consider the **time transformation**

$$t(\tau) = t_0 + \int_{t_0}^{\tau} \omega(r) dr, \quad \tau \in \mathcal{T}$$

with  $\omega(r) \geq 0$  and  $\int_{t_0}^{t_f} \omega(r) dr = t_f - t_0$ .

# Variable Time Transformation

(Dubovitskii, Milyutin, 1963; Tartar, 1979; Gerdts et al., 2012)

The dynamics after transformation read

$$E(y) = \omega(r) \cdot f(t(\tau), y(\tau), \hat{v}(\tau)).$$

State constraints enforce

$$\int_{t_i}^{t_{i+1}} \omega(r) dr = t_{i+1} - t_i = h, \quad 1 \leq i \leq N.$$

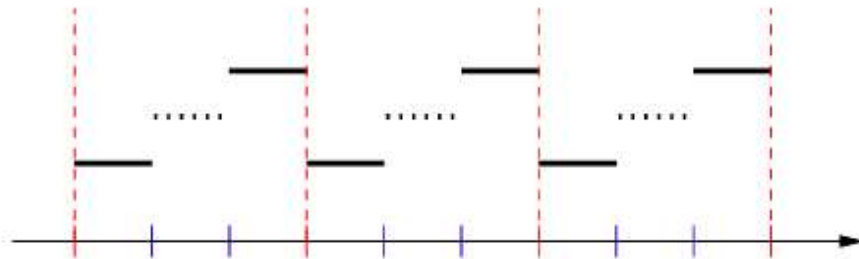
Then  $\omega(r)$  represent the speed of running through a major interval  $[t_i, t_{i+1}]$  and may be chosen differently (by way of optimization) for each minor interval  $1 \leq j \leq M$  associated with mode choice  $v^j$ .

Extremals  $\omega(r) = M$  on  $[\tau_{i,j}, \tau_{i,j+1}]$  (with length  $h/M$ ) for some  $1 \leq j \leq M$ ,  $\omega(r) = 0$  on  $[\tau_{i,k}, \tau_{i,k+1}]$  for all  $k \neq j$  correspond to feasible points of (MIOCP).

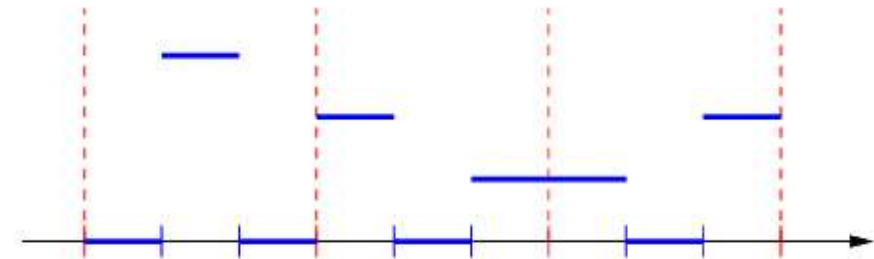
# Variable Time Transformation

(Picture adapted from Gerds, Sager, 2012)

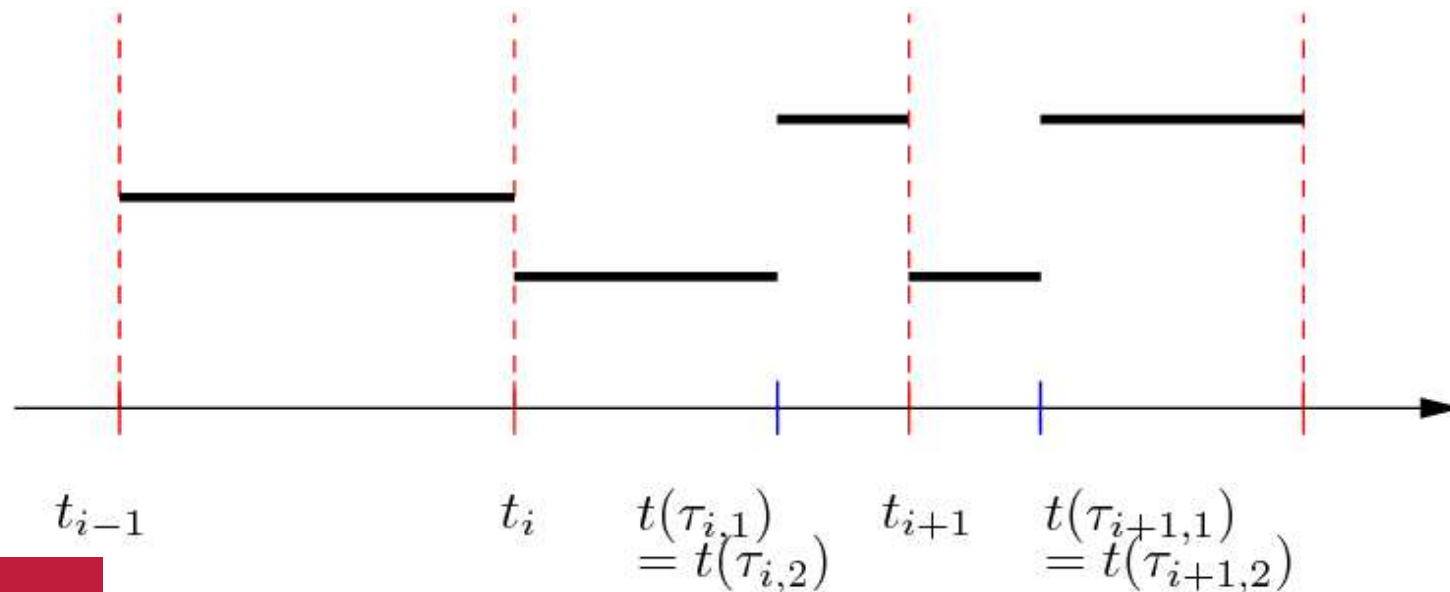
$v(\tau)$ :



$w(\tau)$ :



Corresponding control  $v(t) = v(\tau(t))$ :



# Convexification

(Sager et al., 2005, 09, 12; Vasudevan et al., 2013)

Introduce **binary indicators**  $\omega_i(t, x)$ ,  $1 \leq i \leq M$  to form of a **one-hot encoding** of the discrete choices  $\{v^1, \dots, v^M\}$ :

$$v(t, x) = v^i \iff \omega_i(t, x) = 1, \omega_j(t, x) = 0 \text{ for all } j \neq i.$$

Integer control in terms of  $\omega$ :

$$v(t, x) := \sum_{i=1}^M v^i \cdot \omega_i(t, x).$$

Differential equation in terms of  $\omega$ :

$$E(y) = f(y(t, x), v(t, x)) = \sum_{i=1}^M \omega_i(t, x) \cdot f(y(t, x), v^i).$$

Constraints in terms of  $\omega$ :

$$0 \leq c(y(t, x), v(t, x)) \iff 0 \leq \omega_i(t, x) \cdot c(y(t, x), v^i) \quad \forall 1 \leq i \leq M.$$

# Partially Convexified Problem

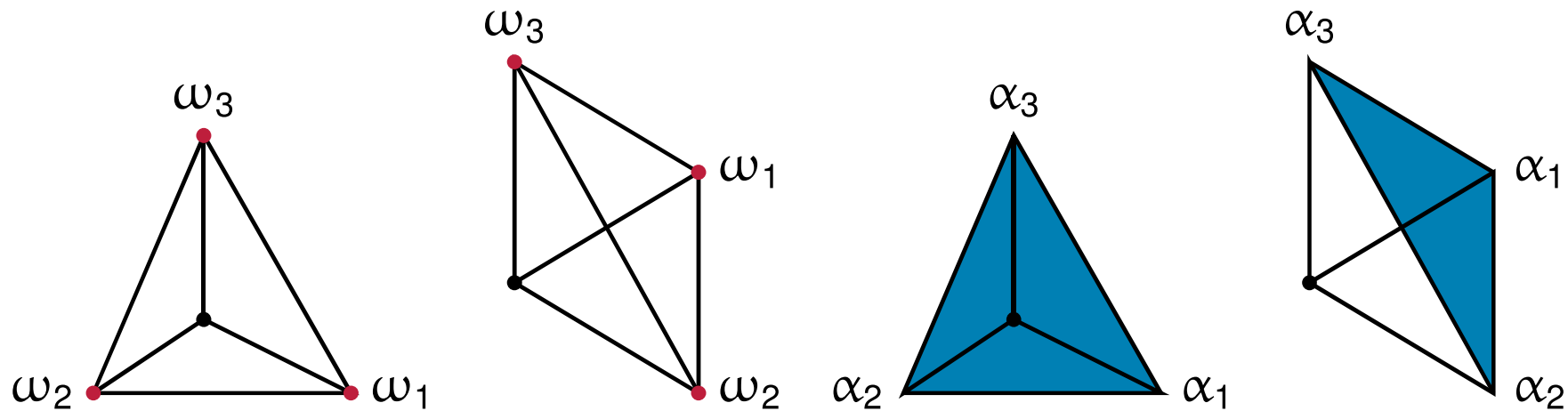
$$\text{(BC)} \quad \left\{ \begin{array}{ll} \inf_{y, \omega} & J(y) + C(\omega) \\ \text{s.t.} & E(y) = \sum_{i=1}^M \omega_i \cdot f(y; v^i) \\ & 0 \leq \omega_i \cdot c(y; v^i) \quad \text{a.e. } (t, x) \in T \times \Omega, \quad 1 \leq i \leq M \\ & \omega(t, x) \in \{0, 1\}^M \quad \text{a.e. } (t, x) \in T \times \Omega \\ & 1 = \sum_{i=1}^M \omega_i(t, x) \quad \text{a.e. } (t, x) \in T \times \Omega \end{array} \right.$$

**Lemma:** There is a bijection between the feasible points of (MIOCP) and those of (BC).

# Relaxation

Replace the **one-hot encoding** by the **set of all convex combinations** to obtain a continuous relaxation (RC).

Geometry: The  $\alpha(t, x)$  can live on the whole of the  $M$ -simplex, not only at its extremal points.



$M = 3$ ,  $(t, x)$  arbitrary but fixed

# Relaxed Problem

$$(RC) \quad \left\{ \begin{array}{ll} \min_{y, \alpha} J(y) + \cancel{C(\alpha)} \\ \text{s.t.} & E(y) = \sum_{i=1}^M \alpha_i \cdot f(y; v^i) \\ & 0 \leq \alpha_i \cdot c(y; v^i) \quad \text{a.e. } (t, x) \in T \times \Omega, \quad 1 \leq i \leq M \\ & \alpha(t, x) \in [0, 1]^M \quad \text{a.e. } (t, x) \in T \times \Omega \\ & 1 = \sum_{i=1}^M \alpha_i(t, x) \quad \text{a.e. } (t, x) \in T \times \Omega \end{array} \right.$$

(RC) is a large-scale, structured, nonlinear programming problem. May violate standard CQs. IPOPT often an excellent choice.

Assumption: (RC) can be solved efficiently.

# Integer Control: Feasibility Relaxation

(Cesari, 1983)

When reachability conditions are violated after fixing binaries, the feasibility of mixed state-control constraints in (MIOCP) or (MIPDECO) usually must be relaxed by  $\delta \geq 0$  (arbitrarily small).

$$(BC_\delta) \quad \left\{ \begin{array}{ll} \inf_{y, u, \omega} & J(y) + C(\omega) \\ \text{s.t.} & E(y) = \sum_{i=1}^M \omega_i \cdot f(y; v_i) \\ & -\delta \leq \omega_i \cdot c(y; v_i) \quad \text{a.e. } (t, x) \in T \times \Omega, \quad 1 \leq i \leq M \\ & \omega(t, x) \in \{0, 1\}^M \quad \text{a.e. } (t, x) \in T \times \Omega \\ & 1 = \sum_{i=1}^M \omega_i(t, x) \quad \text{a.e. } (t, x) \in T \times \Omega \end{array} \right.$$

# Intermission

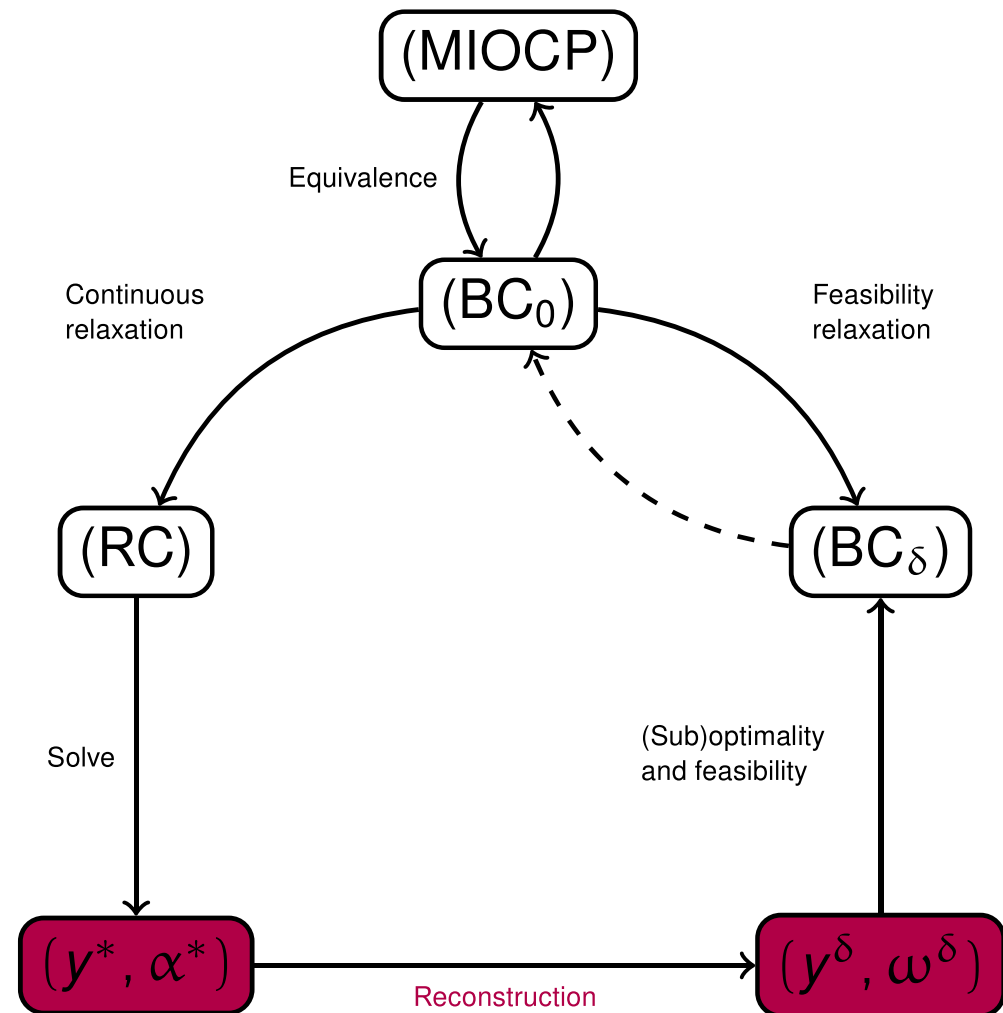
We now have

- $(MIOCP) \equiv (BC_0)$
- a feasibility relaxation  $(BC_\delta)$
- a continuous relaxation  $(RC)$

We can solve  $(RC)$  to obtain  $\alpha^*$  and a state  $y^* = y(\alpha^*)$ .

**Goals:** From these, find  $\omega^\delta$  and  $y^\delta = y(\omega^\delta)$  for  $(BC_\delta)$  and achieve  $\delta \rightarrow 0$ .

Relate  $\alpha^*$  to  $\omega^\delta$  and  $y^*$  to  $y^\delta$ .



# State Perturbation

Motivating ODE example: Let  $x, y \in W^{1,\infty}(\mathcal{T}, \mathbb{R}^n)$  and  $\alpha, \beta \in [0, 1]^M$ , and consider the initial value problems

$$\begin{cases} \dot{y}(t) = f_0(x(t)) + F(x(t)) \alpha(t), & y(0) = y^0 & t \in \mathcal{T}, \\ \dot{\bar{y}}(t) = f_0(\bar{y}(t)) + F(\bar{y}(t)) \beta(t), & \bar{y}(0) = y^0 & t \in \mathcal{T}. \end{cases}$$

Assume that  $t \mapsto F(y(t))$  is continuously differentiable and that there are constants  $M_1, M_2 > 0$  and  $\delta > 0$  such that

$$\left\| \int_0^t \alpha(s) - \beta(s) \, ds \right\| \leq \delta, \quad \|F(y(t))\| \leq M_1, \quad \left\| \frac{d}{dt} F(y(t)) \right\| \leq M_2, \quad t \in \mathcal{T}.$$

Then  $\|y(t) - \bar{y}(t)\| \leq \delta(M_1 + tM_2) \exp((L_1 + L_2)t)$ .

**Bottom line:** As  $\delta \rightarrow 0$  or  $t \rightarrow 0$  the approximation becomes arbitrarily tight in state space.

# Chain of Approximation Arguments

1. Solve (RC) for optimal values  $\alpha^*$ ,  $y^*$ . For a grid  $h$ , obtain  $\omega^h$  from  $\alpha^*$ .
2. Show for a pseudo-metric  $d$  that

$$h \rightarrow 0 \text{ implies } d(\omega^h, \alpha^*) \rightarrow 0.$$

3. A condition on the grids gives approximation of  $\alpha^*$  by  $\omega^h$  in a suitable topology,  
$$\omega^h \rightharpoonup \alpha^*.$$

4. A suitable property of the solution operator  $S$  of the state equation yields that

$$y^h = y(\omega^h) \rightarrow y(\alpha^*) = y^*$$

in norm.

5. Continuity of  $J$  in  $y$  yields

$$J(y^h) \rightarrow \min_{(RC)} J(y).$$

6. If constraints  $0 \leq c(\dots, v(t, x))$  are absent from (MIOCP), we have

$$\min_{(RC)} J(y) = \inf_{(BC_0)} J(y).$$



# Infimizers

$(BC_\delta)$  need not have a proper minimizers  $\omega^*$  in  $L^\infty(\mathcal{T}, \{0, 1\}^M)$ .

In general, only limiting sequences  $(\omega^n)^n$  exist that convergence in a weak topology and with a weak limit in  $L^\infty(\mathcal{T}, [0, 1]^M)$ .

Example:

$$\omega^n(t) = \begin{cases} (1, 0)^T & \text{if } t \in [\frac{2i}{n}, \frac{2i+1}{n}] \text{ for some } 0 \leq i < n/2 \\ (0, 1)^T & \text{otherwise} \end{cases}$$

is binary feasible and  $\omega^n(t) \in L^\infty([0, 1], \mathbb{R}^2)$ , but the limit as  $n \rightarrow \infty$  is not.

However,

$$\int_0^t \omega^n(s) \, ds \rightarrow \int_0^t \left(\frac{1}{2}, \frac{1}{2}\right)^T \, ds \quad \text{a.e. } t \in \mathcal{T} \quad \text{as } n \rightarrow \infty.$$

Elements of the limiting sequence approximate a relaxed control arbitrarily well.

# Historical works

The approximation methodology can already be found in:

- Lyapunov's convexity theorem Lyapunov, 1940
- Filippov-Ważewski theorem Filippov, 1958, Ważewski, 1963
- Lindenstrauss theorem Lindenstrauss, 1966
- Tartar's work Tartar, 1979

Algorithms & application to (mixed)-integer optimal control is what's newer.

- Construct weakly converging approximating control sequences  $(\alpha^n)_n$
- Rates of convergence on various domains
- Restrictions on switch counts, dwell times, etc.

# Integrality Gap Pseudometric

Let  $(\phi^n)^n \subset L^\infty((0, T), \mathbb{R})$  be a bounded sequence of controls such that  $\Phi^n(t) := \int_0^t \phi^n(s) \, ds$  satisfies

$$\|\Phi^n\|_{L^\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then, we call  $(\phi^n)^n$  a sequence of **vanishing integrality gap**.

For a discretization grid  $\{t_i \mid 0 \leq i \leq N\}$  and choices  $1 \leq j \leq M$ , let

$$\phi_j^n(s) := \alpha_j(s) - \omega_{i,j}^n$$

to find that, per choice  $j$ ,

$$\|\Phi_j^n\|_{L^\infty} = \max_{0 \leq i < N} \left| \int_0^{t_i} \alpha_j(s) - \omega_{i,j}^n \, ds \right|.$$

# State Approximation

(Sager et al., 2009, 12; Manns, K., 2020)

Let  $\alpha \in L^\infty((0, T), \mathbb{R}^M)$  such that  $\|\alpha\|_{L^\infty} \leq 1$ ,  $(\omega^n)^n$  be binary-valued functions such that the coordinate sequences of  $(\phi^n)^n$  defined by  $\phi^n := \alpha - \omega^n$  are of **vanishing integrality gap**.

Let  $y, y^n$  for  $n \in \mathbb{N}$  be the unique mild solutions of (RC) and  $(BC_\delta)$ .

Furthermore, let  $f_i(t) := f(t, y(t), v_i)$  be in  $L^1((0, T), X)$  for  $1 \leq i \leq M$ .

Then,

$$\|y - y^n\|_{C([0, T], X)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

i.e., the  $y_n$  converge to  $y$  as continuous functions.

# Objective Approximation

(Sager et al., 2009, 12; Manns, K., 2020)

Let  $(\bar{y}, \bar{\alpha})$  be feasible for (RC) such that  $\bar{y} = y(\bar{\alpha})$  is the unique mild solution.

Let  $[0, T] \ni s \mapsto f(s, y(s), v^i) \in X$  be in  $L^1((0, T), X)$  for  $1 \leq i \leq M$ .

Let  $(\omega^n)^n$  be binary-valued functions such that the coordinate sequences of  $(\phi^n)^n$  defined by  $\phi^n := \bar{\alpha} - \omega^n$  are of vanishing integrality gap.

Then, for every  $\delta > 0$ , there exists  $(y^\delta, \omega^\delta)$  feasible for  $(BC_\delta)$  such that

$$|J(\bar{y}) - J(y^\delta)| < \delta.$$

We call  $(y^\delta, \omega^\delta)$   **$\delta$ -optimal solutions** and can get them for any  $\delta > 0$  as long as we can afford to solve (RC) with an  $N$  large enough.

# Integral Approximation of the Relaxed Control

**Open point:** How to find  $(\omega^n)^n$  such that the integrality gap vanishes,  $\|\Phi^n\|_{L^\infty} \rightarrow 0$  as  $n \rightarrow \infty$  ?

(Sager, 2005)

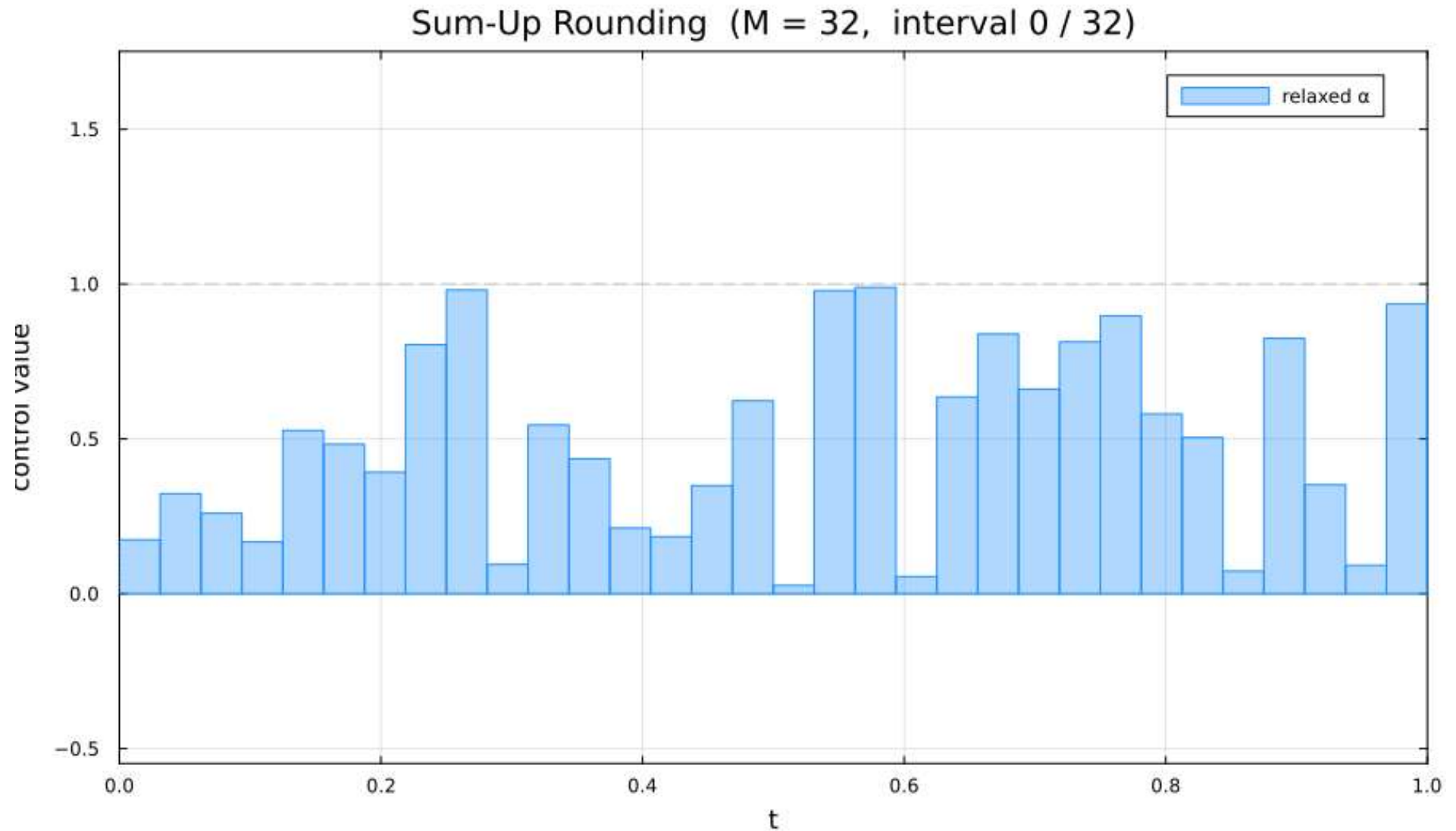
An early algorithm was **Sum-Up Rounding** (SUR): Iteratively choose  $\omega$  to keep the entries of  $(\phi^n)^n$  balanced:

1. Let  $t_0 < \dots < t_N = t_f$  discretize  $[t_0, t_f]$ . Let  $\alpha$  be a relaxed control.
2. Define  $\omega(\alpha) : [0, T] \rightarrow \{0, 1\}^M$  iteratively for  $i = 0, \dots, N - 1$  as

$$\omega(\alpha)_j(t)|_{[t_i, t_{i+1}]} := \begin{cases} 1 & : j = \arg \max_{k \in \{1, \dots, M\}} \int_0^{t_{i+1}} \alpha_k - \int_0^{t_i} \omega(\alpha)_k, \\ 0 & : \text{else.} \end{cases}$$

This has  $O(N)$  runtime complexity

# Integral Approximation of the Relaxed Control



# Sum-Up Rounding Approximation Bound

(K. et al, 2020)

Let  $M \geq 2$ , and let  $\bar{\alpha} \in L^\infty(\mathcal{T}, [0, 1]^M)$  be feasible. Let  $\varepsilon > 0$  and let  $\omega$  be the control constructed from  $\bar{\alpha}$  by SUR.

Then  $\omega \in [0, 1]^M$ , and for all  $t \in \mathcal{T}$  we have the estimate

$$\left\| \int_0^t \bar{\alpha}(s) - \omega(s) \, ds \right\|_\infty \leq \bar{\Delta} \cdot c(M),$$

with the bound

$$c(M) := \sum_{m=1}^{N-1} \frac{1}{M+1-m} \in O(\log M).$$

This bound is tight.

# SUR with mixed state-control constraints

(K., 2010)

Algorithm SUR-VC:

1. Let  $t_0 < \dots < t_N = t_f$  discretize  $[t_0, t_f]$ . Let  $\alpha$  be a relaxed control.
2. Define  $\omega(\alpha) : [0, T] \rightarrow \{0, 1\}^M$  iteratively for  $i = 0, \dots, N - 1$  as

$$\omega(\alpha)_j(t)|_{[t_i, t_{i+1}]} := \begin{cases} 1 & : j = \arg \max_{k \in \mathcal{F}_i} \int_0^{t_{i+1}} \alpha_k - \int_0^{t_i} \omega(\alpha)_k, \\ 0 & : \text{else.} \end{cases}$$

where the sets  $\mathcal{F}_i \subseteq \{1, \dots, M\}$  are defined by

$$\mathcal{F}_i := \left\{ 1 \leq j \leq M \mid \int_{t_i}^{t_{i+1}} \alpha_j(t) dt > 0 \right\}.$$

This ensures feasibility of  $0 \leq c(t, x(t), v^j)$  when  $\omega_j(t) = 1$ .

# SUR-VC Approximation Bound

(Manns, K., 2020)

Let  $M \geq 2$ , and let  $\bar{\alpha} \in L^\infty(\mathcal{T}, [0, 1]^M)$  be feasible. Let  $\varepsilon > 0$  and let  $\omega$  be the control constructed from  $\bar{\alpha}$  by SUR-VC.

Then  $\omega \in [0, 1]^M$ , and for all  $t \in \mathcal{T}$  we have the estimate

$$\left\| \int_0^t \bar{\alpha}(s) - \omega(s) \, ds \right\|_\infty \leq \bar{\Delta} \cdot c(M),$$

with the bound

$$c(M) := \left\lfloor \frac{M-1}{2} \right\rfloor.$$

This bound is tight.

# Order Conserving Domain Dissections

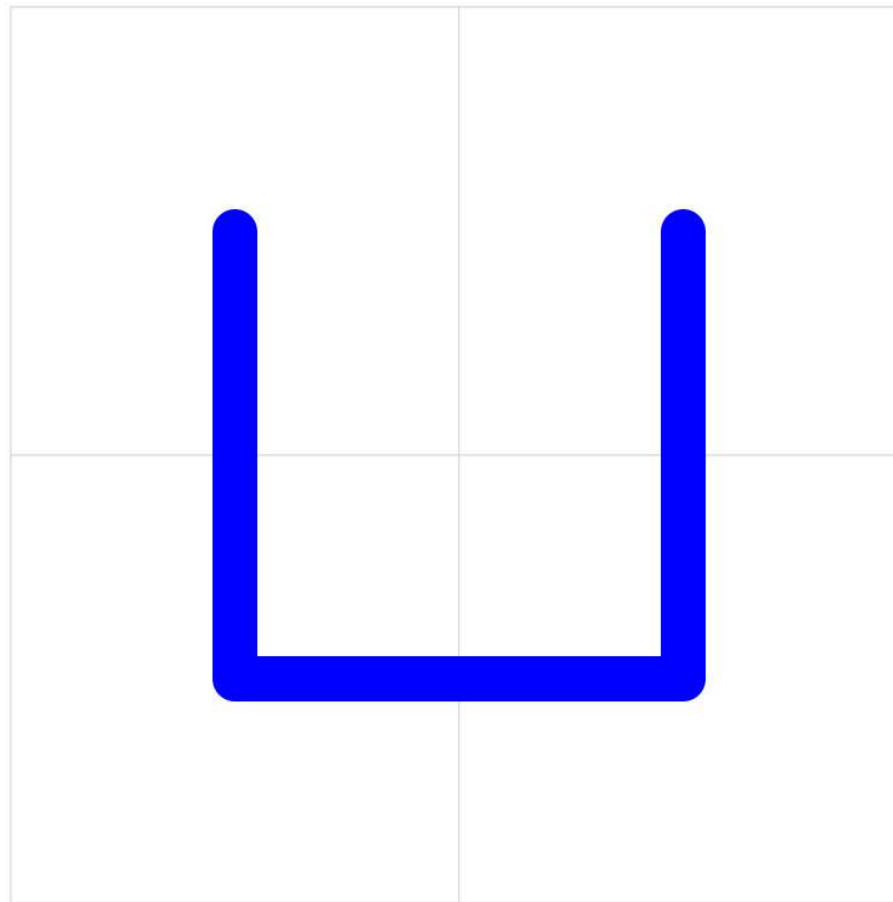
(Manns et al., 2021)

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain. We call  $\left(\left\{S_1^{(n)}, \dots, S_{N^{(n)}}^{(n)}\right\}\right)_n \subset 2^{\mathcal{B}(\Omega)}$  an **order conserving domain dissection** of  $\Omega$  if

1.  $N^{(0)} = 1, S_1^{(0)} = \Omega$
2.  $\bigcup_{i=1}^{N^{(n)}} S_i^{(n)} \subset \Omega, S_i^{(n)} \cap S_j^{(n)} \neq \emptyset$  for all  $i \neq j$  and  $\lambda\left(\bigcup_{i=1}^{N^{(n)}} S_i^{(n)}\right) = \lambda(\Omega)$ , i.e. the  $S_i^{(n)}$  form a disjoint decomposition of the domain except for possible zero sets.
3. For all  $n \in \mathbb{N}$  and all  $i \in \{1, \dots, N^{(n)}\}$ ,  $\lambda(S_i^{(n)}) > 0$
4. For all  $n \in \mathbb{N}$  and all  $i \in \{1, \dots, N^{(n-1)}\}$ , there exists  $j, k \in \{1, \dots, N^{(n)}\}, j < k$  such that  $\bigcup_{l=j}^k S_l^{(n)} \subset S_i^{(n-1)}$  and  $\lambda\left(\bigcup_{l=j}^k S_l^{(n)}\right) = \lambda(S_i^{(n-1)})$ , i.e. the order of the discretization cells is preserved from one iteration to the next.
5.  $\max_{i \in \{1, \dots, N^{(n)}\}} \lambda(S_i^{(n)}) \rightarrow 0$ , i.e. the maximum cell size goes to zero.
6. The  $\sigma$ -algebra generated by  $\bigcup_{n=1}^{\infty} \left\{S_1^{(n)}, \dots, S_{N^{(n)}}^{(n)}\right\}$  is  $\mathcal{B}(\Omega)$ .

# A space-filling curve hierarchy

(D Hilbert, 1891)



# Serialization of Sum-Up Rounding

- Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain. Let  $\{S_1, \dots, S_N\} \subset \mathcal{B}(\Omega)$  be a finite partition of  $\Omega$ .
- For  $i \in \{1, \dots, N\}$ , we define iteratively

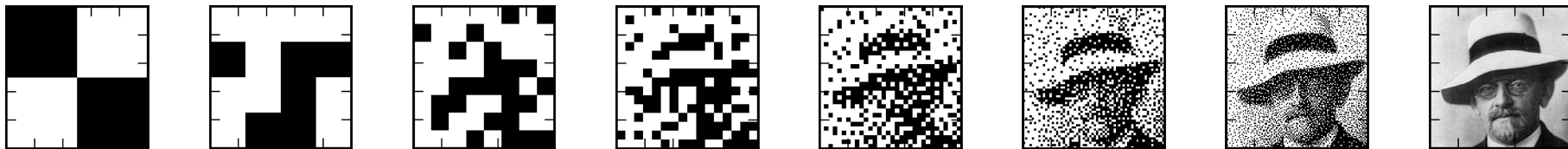
$$\omega(\alpha)_j|_{S_i} := \begin{cases} 1 : & j = \arg \max_{k \in \{1, \dots, M\}} \int_{S_i} \alpha_k \, d\lambda + \int_{\bigcup_{\ell=1}^{i-1} S_\ell} \alpha_k - \omega(\alpha)_k \, d\lambda \\ 0 : & \text{otherwise} \end{cases}$$

# Visualizing convergence

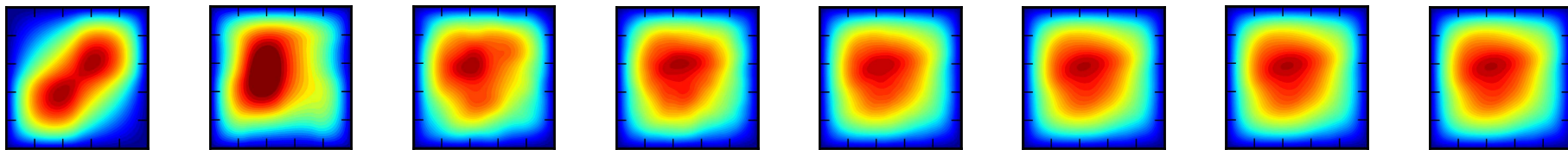
Example: approximation  $d(\omega^h, \alpha) \rightarrow 0 \Rightarrow \omega^h \rightharpoonup \alpha \Rightarrow y(\omega^h) \rightarrow y(\alpha)$

$$-\Delta y = \alpha, \quad y|_{\partial\Omega} = 0, \quad -\Delta y^n = \omega^n, \quad \omega^n|_{\partial\Omega} = 0$$

Weak(\*) convergence in control space:  $\omega^n \rightharpoonup \alpha$



Norm convergence in state space:  $y(\omega^n) \rightarrow y(\alpha)$



# Integral Approximation of the Relaxed Control

K., Manns, Ulbrich, 2019

Let an order conserving domain dissection be given, and let the ratio between the maximum and minimum diameters of the grid cells be uniformly bounded over the grid iterations. Let  $H^d \leq \bar{\Delta}^{(n)}$  hold if  $H$  is the diameter of a grid cell in iteration  $n$ .

Let  $(\omega^{(n)})_n$  be a sequence of binary controls such that for the  $n$ -th rounding grid, the function  $\omega^{(n)}$  is constant per grid cell.

Furthermore, let  $d^{(n)}(\omega^{(n)}, \alpha) \leq C_4 \bar{\Delta}^{(n)}$  for some  $C_4 > 0$ .

Then, for all  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  and  $C(\varepsilon) > 0$  such that for all  $n \geq n_0$ , we obtain the estimate

$$\|\alpha - \omega^{(n)}\|_{H^{-1}(\Omega_T)} \leq C(\varepsilon) (\bar{\Delta}^{(n)})^{\frac{1}{1.5d + 0.5d^2 + \varepsilon 1_{d=2}}}.$$

The constant  $C(\varepsilon)$  only depends on  $\varepsilon$  if  $d = 2$ .

# Mixed-Integer PDE Example in 2D

(Leyffer, Manns et al., 2020)

Minimize superposition of incoming wave  $u_0$  and response  $u$

$$\inf_{u,v} \quad \frac{1}{2} \|u + u_0\|_{L^2(D_0)}^2$$

Helmholtz wave equation for response

$$\text{s.t. } -\Delta u - k_0^2(1 + qv) u = k_0^2 qv u_0 \quad \text{on } \mathcal{D}$$

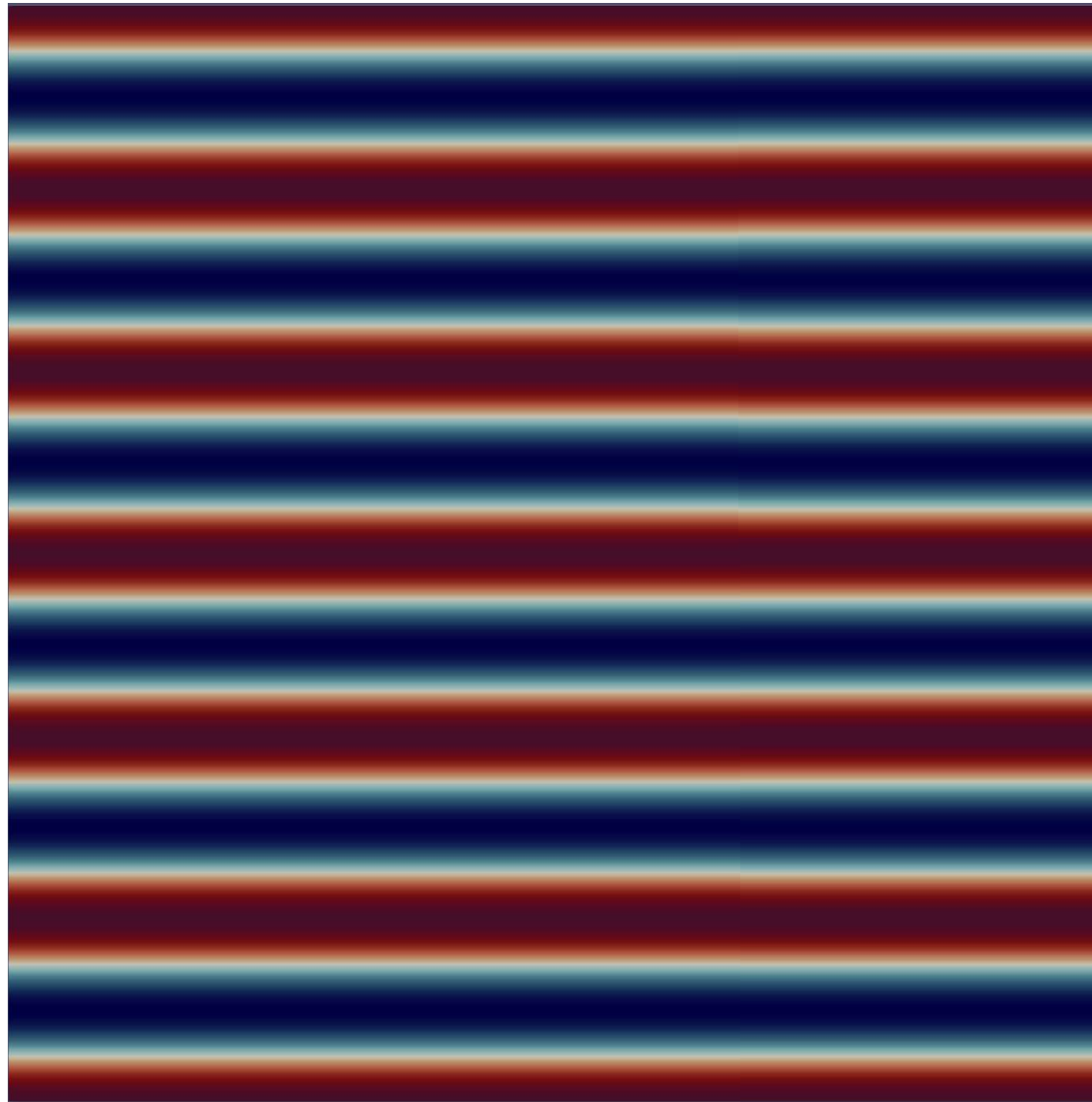
Absorbing Robin boundary condition

$$\frac{\partial u}{\partial n} - \hat{\imath} k_0 u = 0 \quad \text{on } \partial D$$

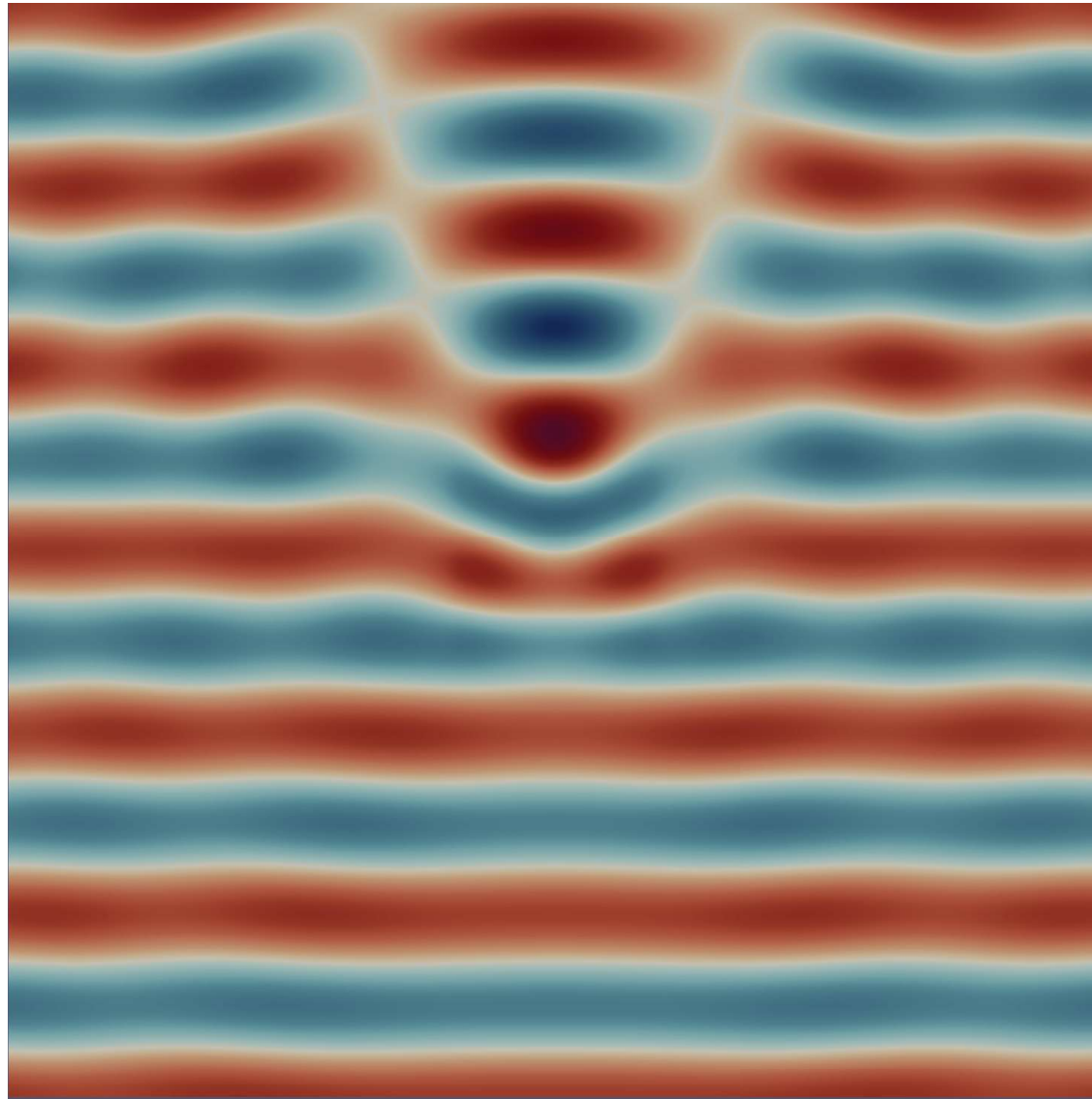
Discrete choice of material affects response

$$\begin{aligned} v(x) &\in \{v^1, \dots, v^M\} \quad \text{on } \mathcal{D}_c \\ v(x) &= 0 \quad \text{on } \mathcal{D} \setminus \mathcal{D}_c \end{aligned}$$

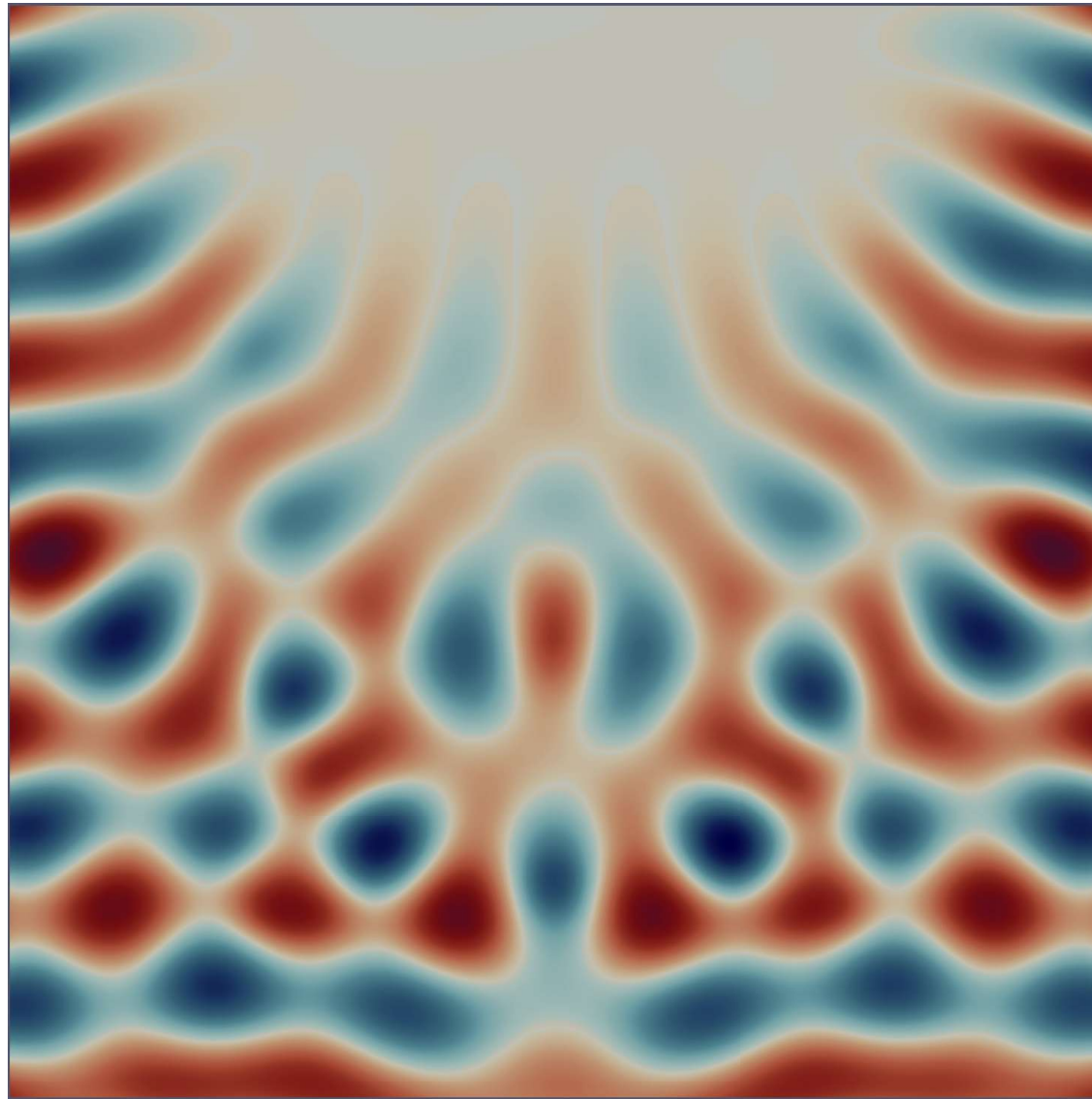
# Mixed-Integer PDE Example in 2D



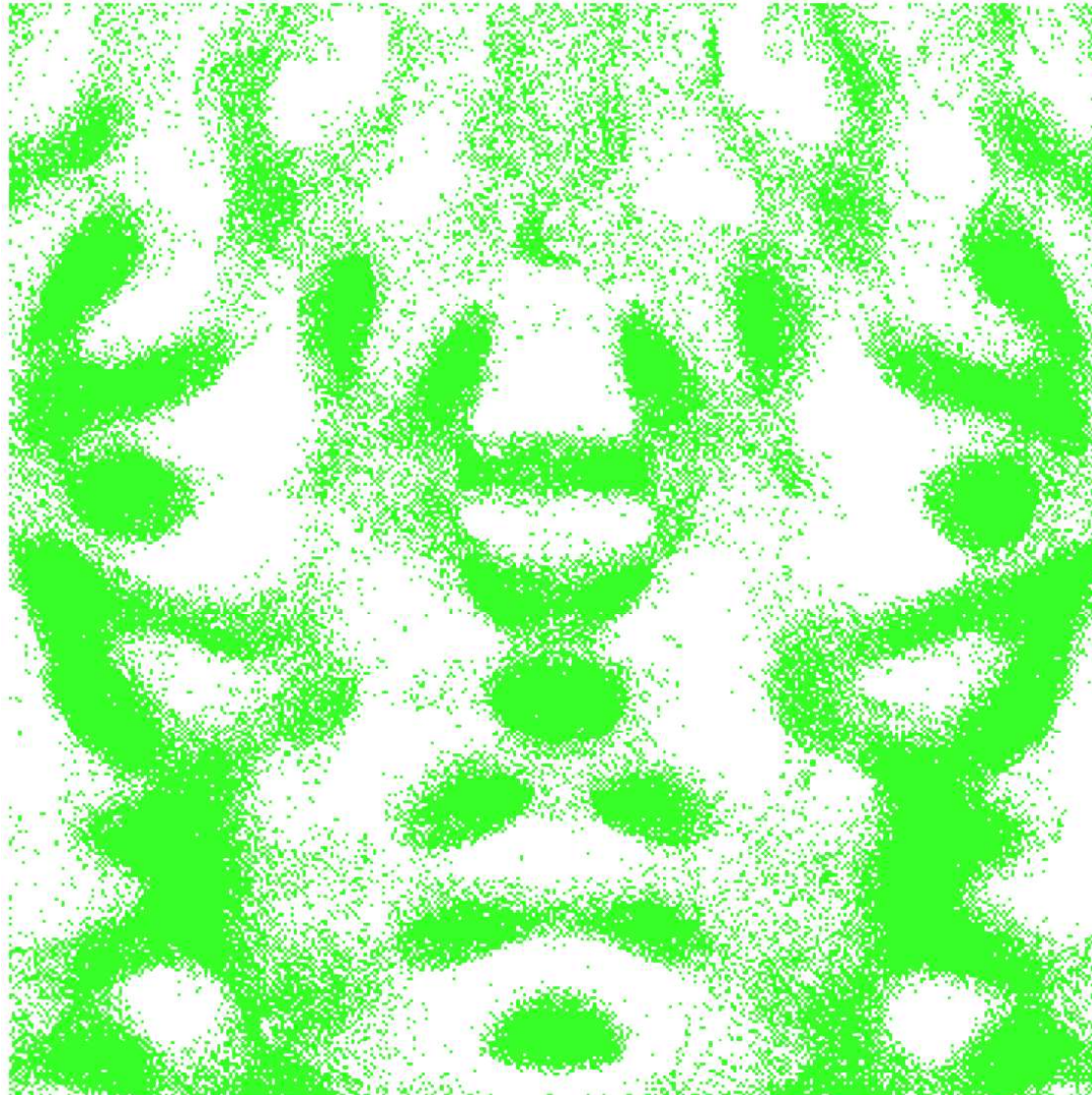
# Mixed-Integer PDE Example in 2D



# Mixed-Integer PDE Example in 2D



# Mixed-Integer PDE Example in 2D



# Switch Costs and Restrictions

- State-space effect of rounding
- Attribute costs to staying in mode  $1 \leq j \leq M$
- Attribute costs to entering mode  $1 \leq j \leq M$  initially or from another mode
- Remain in mode  $1 \leq j \leq M$  for a minimum time if entered (min-up)
- Stay away from mode  $1 \leq j \leq M$  for a minimum time if entered (min-down)

Some approaches:

- Alternatingly approximate  $(BC_\delta)$  and solve for  $v$  only (Hante, Potschka et al, 2021)
- Minimize integrality gap plus costs subject to constraints  
—→ combinatorial integral approximation (CIA) (Jung et al., 2011; Zeile, Sager et al., 2020)
- Minimize costs subject to SUR integrality gap and constraints  
—→ switch-cost aware rounding problems (SCARP) (Bestehorn et al., 2020)
- Dwell times/Switch interdictions  
—→ rounding-via-matchings (RVM) (Bestehorn et al., 2021)

# Switch Cost Aware Rounding Procedure (SCARP)

(Bestehorn et al., 2021)

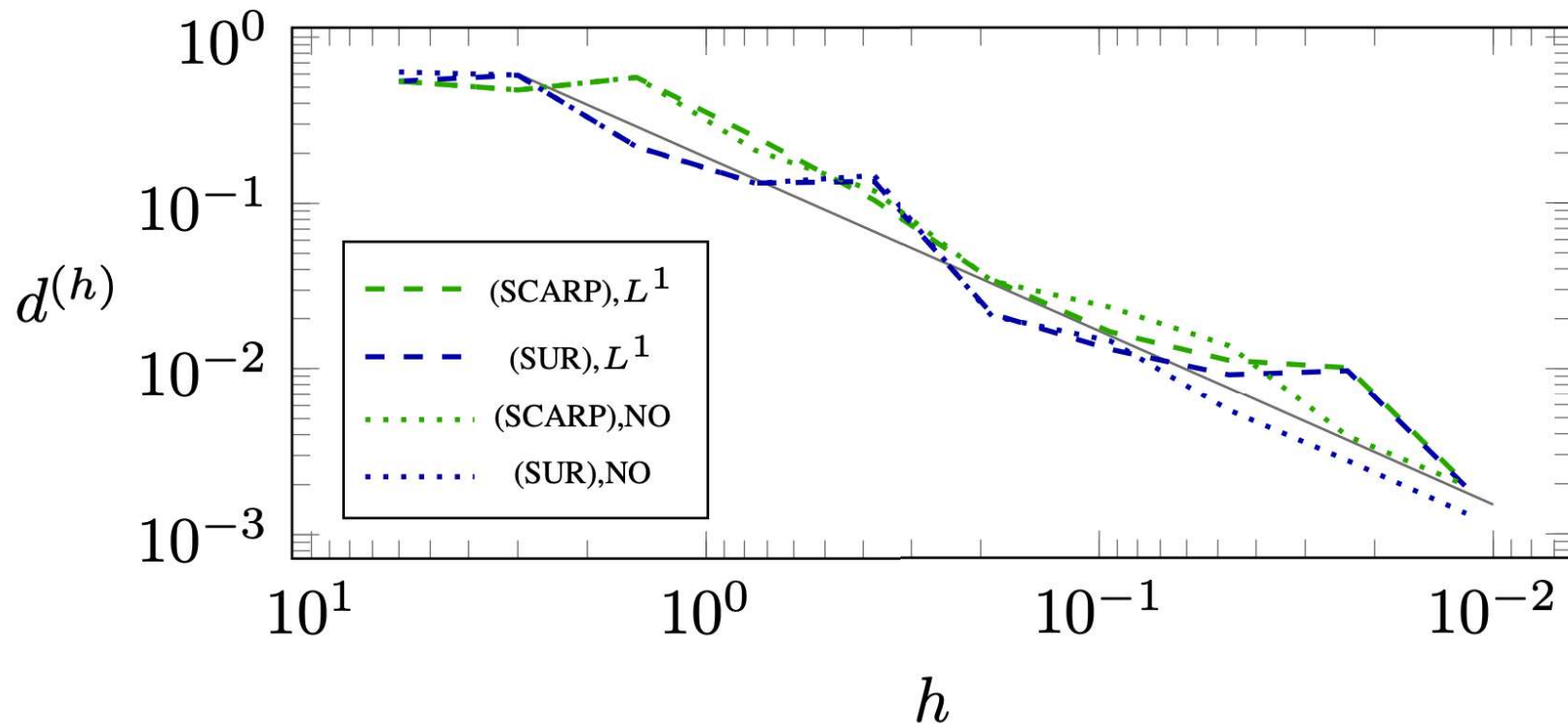
$$\begin{aligned}
 & \min_{\omega_{k,i}, \epsilon_{k,i}, \lambda_{k,i}} \sum_{k=1}^{N-1} \sum_{i=1}^M c_i \epsilon_{k,i} + d_i \lambda_{k,i} \\
 & \quad + \sum_{i=1}^M c_i \omega_{1,i} + \sum_{i=1}^M d_i \omega_{N,i} \\
 & \text{s.t. } \sum_{i=1}^M \omega_{k,i} = 1 \text{ for all } k \in \{1, \dots, N\} \\
 & \quad -Kh \leq \sum_{\ell=1}^k h_{\ell} (\alpha_{\ell,i} - \omega_{\ell,i}) \leq Kh \quad \text{for all } k \in \{1, \dots, N\}, i \in \{1, \dots, M\} \\
 & \quad \omega_{k+1,i} - \omega_{k,i} \leq \epsilon_{k,i} \quad \text{for all } k \in \{1, \dots, N-1\}, i \in \{1, \dots, M\} \\
 & \quad \omega_{k,i} - \omega_{k+1,i} \leq \lambda_{k,i} \quad \text{for all } k \in \{1, \dots, N-1\}, i \in \{1, \dots, M\} \\
 & \quad \omega_{k,i}, \epsilon_{k,i}, \lambda_{k,i} \in \{0, 1\} \text{ for all } i, k
 \end{aligned}$$

SUR and SUR-VC produce feasible points / upper bounds.

# Reconstruction via MILP

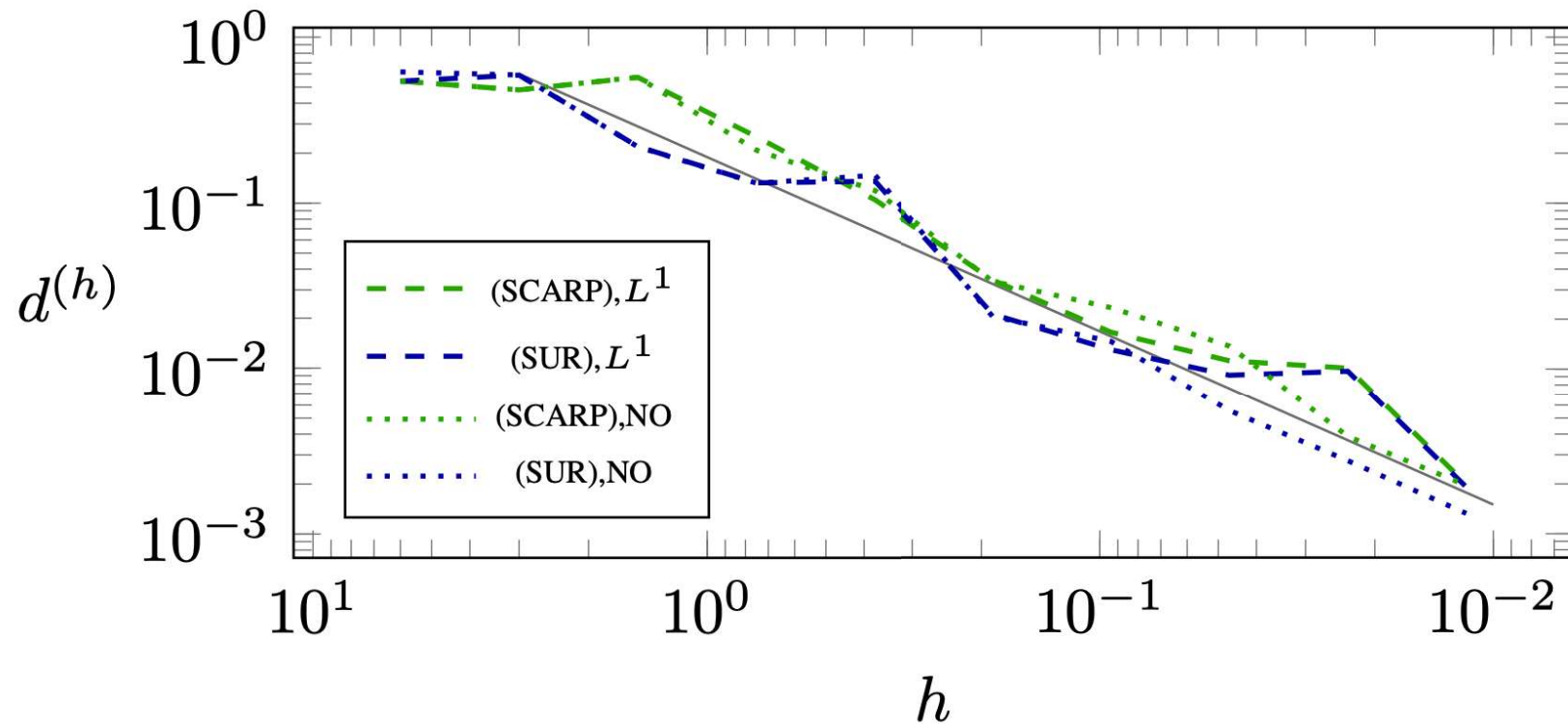
$$d^{(h)} = \frac{\sup_t \|\mathbf{y}(\omega^{(h)}(t)) - \mathbf{y}(\alpha)(t)\|_2}{\sup_t \|\mathbf{y}(\alpha)(t)\|_2}.$$

Control approximation (here  $K = 5/6$ ):



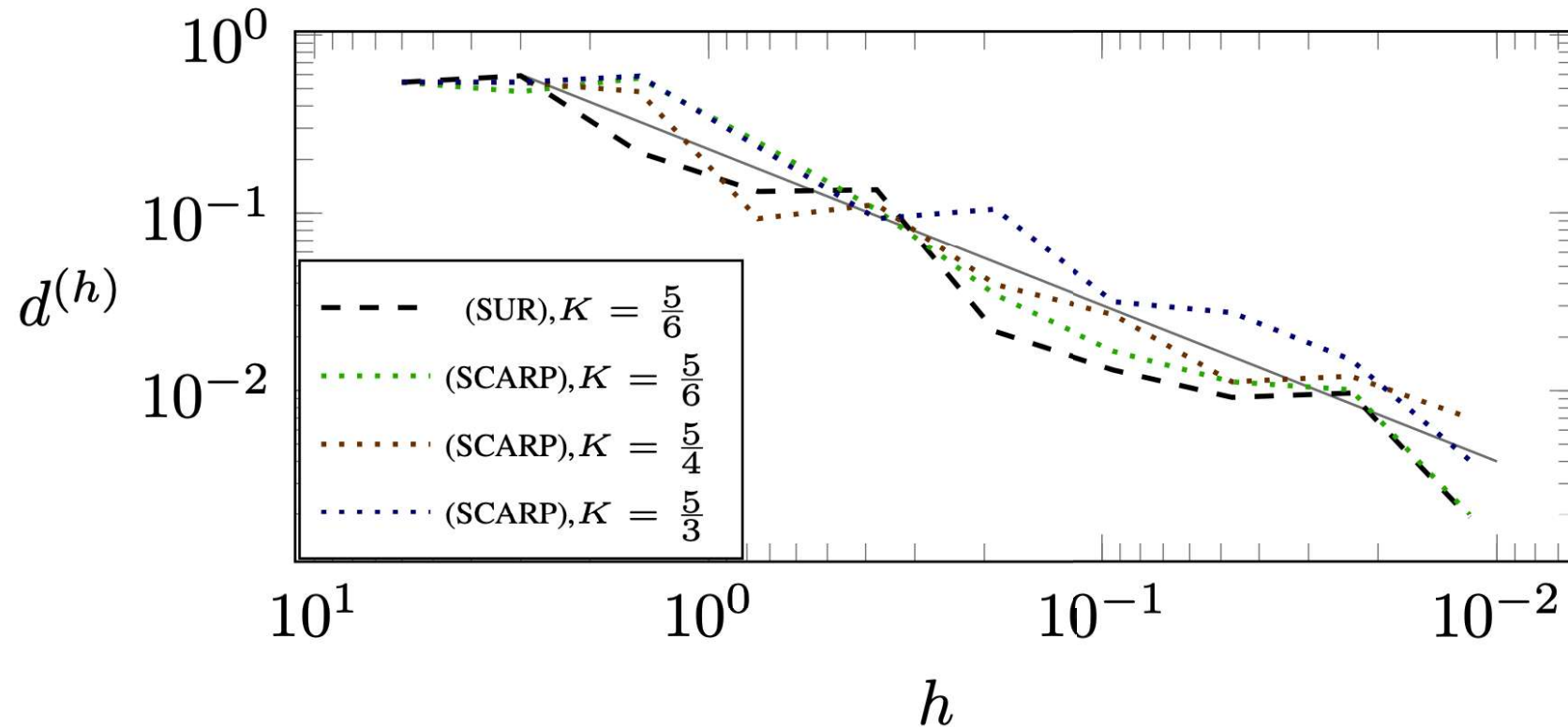
# Reconstruction via MILP

Objective function values (here  $K = 5/6$ ):



# Reconstruction via MILP

State vector approximation:



# Rounding via Shortest Paths

(Bestehorn et al., 2021)

Can model MILP as shortest path problem on a DAG. Accelerated Dijkstra's algorithm gives huge speedups over generic MILP solvers.

Vertex set

$$V := V(\alpha, \theta) = \bigcup_{t=1}^N V_t \text{ with } V_t := V_t(\alpha, \theta),$$

with

$$V_t(\alpha, \theta) := \left\{ \omega \in \{0, 1\}^{t \times M} \left| \begin{array}{l} \omega \text{ satisfies SOS-1 and the SUR bound w.r.t} \\ \alpha \text{ for all } i \in [M] \text{ and } k \in [t] \end{array} \right. \right\}.$$

Arc set

$$A := \left\{ (\omega^t, \omega^{t+1}) \in V \times V \left| \omega^t \in V_t, \omega^{t+1} \in V_{t+1}, \omega_k^{t+1} = \omega_k^t \text{ for all } k \in [t] \right. \right\}.$$

Costs for  $t \in [N - 1]$  of an arc  $(v, w) \in V_t \times V_{t+1} \cap A$

$$\overline{C}(v, w) := c^i(v_t, w_{t+1}) + \begin{cases} c^s(v_1) & \text{if } t = 1, \\ 0 & \text{if } t \in \{2, \dots, N - 2\}, \\ c^f(w_N) & \text{if } t = N - 1. \end{cases}$$

# Rounding via Shortest Paths

(Bestehorn et al., 2021)

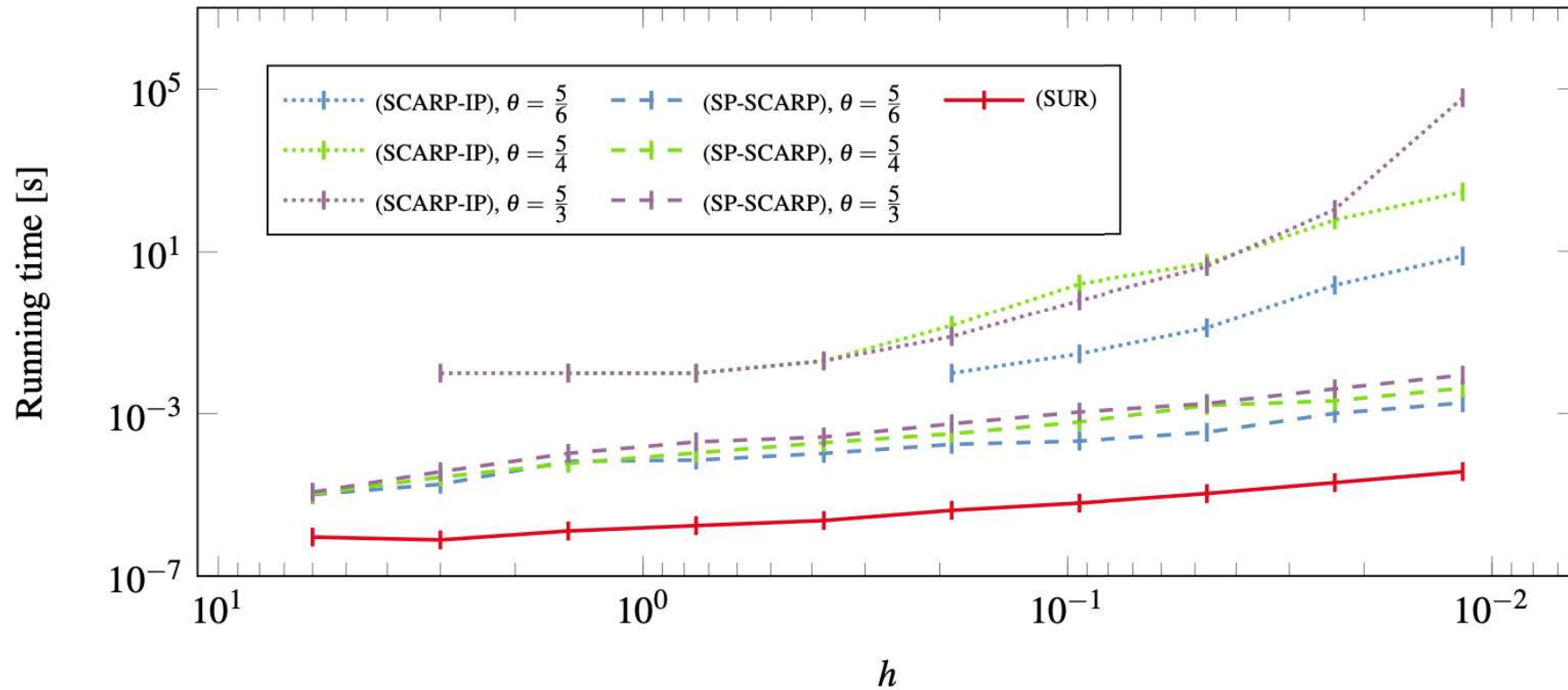
Bounds on  $|V|$  and  $|A|$  can be found by a labelling scheme and counting integer solutions.

The DAG size is bounded by

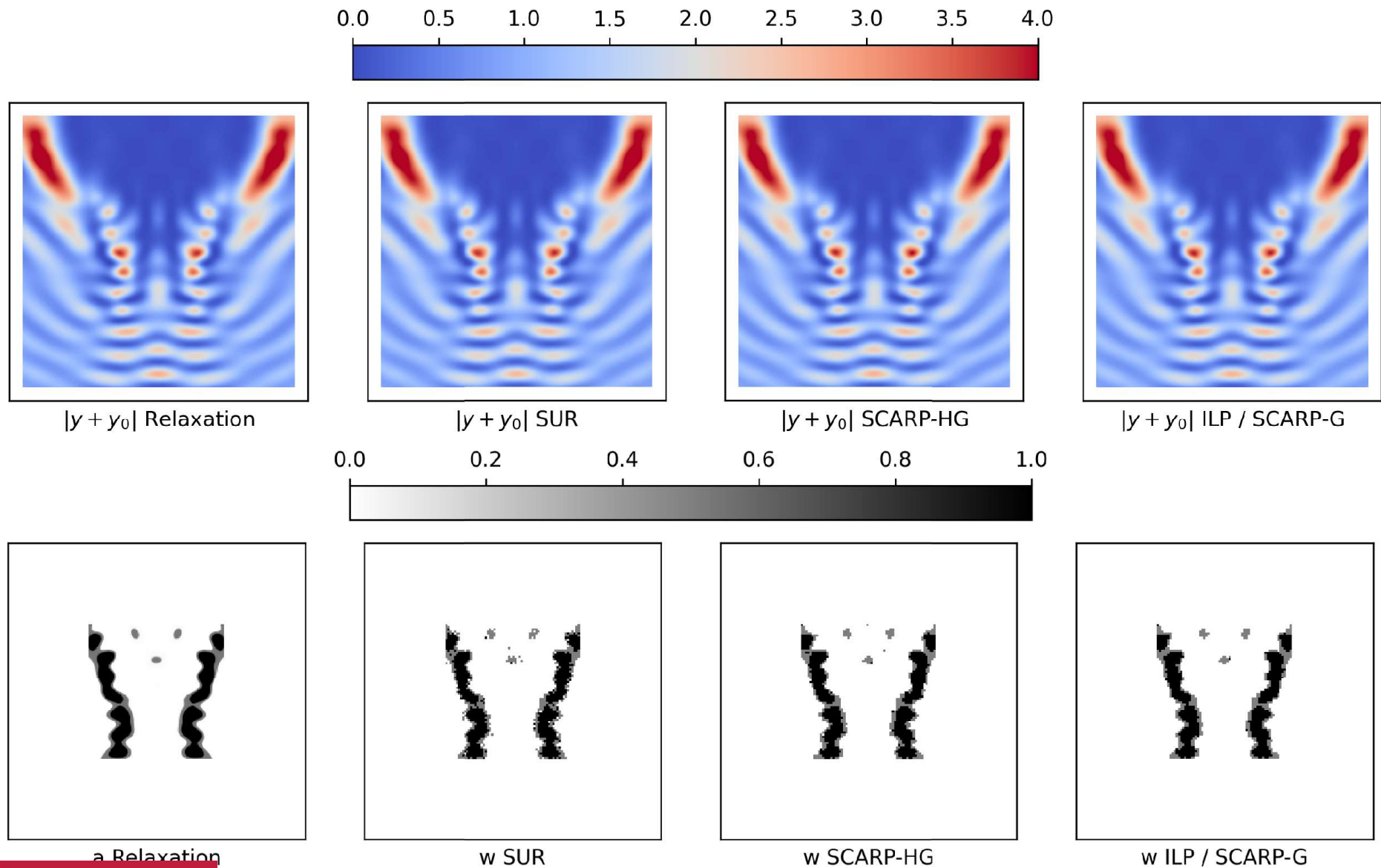
$$|V| + |A| \in O\left(\frac{N(\lfloor 2\theta \rfloor + 3)^{2M}}{M((\lfloor 2\theta \rfloor + 3)^2 - 1)}\right),$$

hence exponential in  $M$  and linear in  $N$  and  $\theta$ .

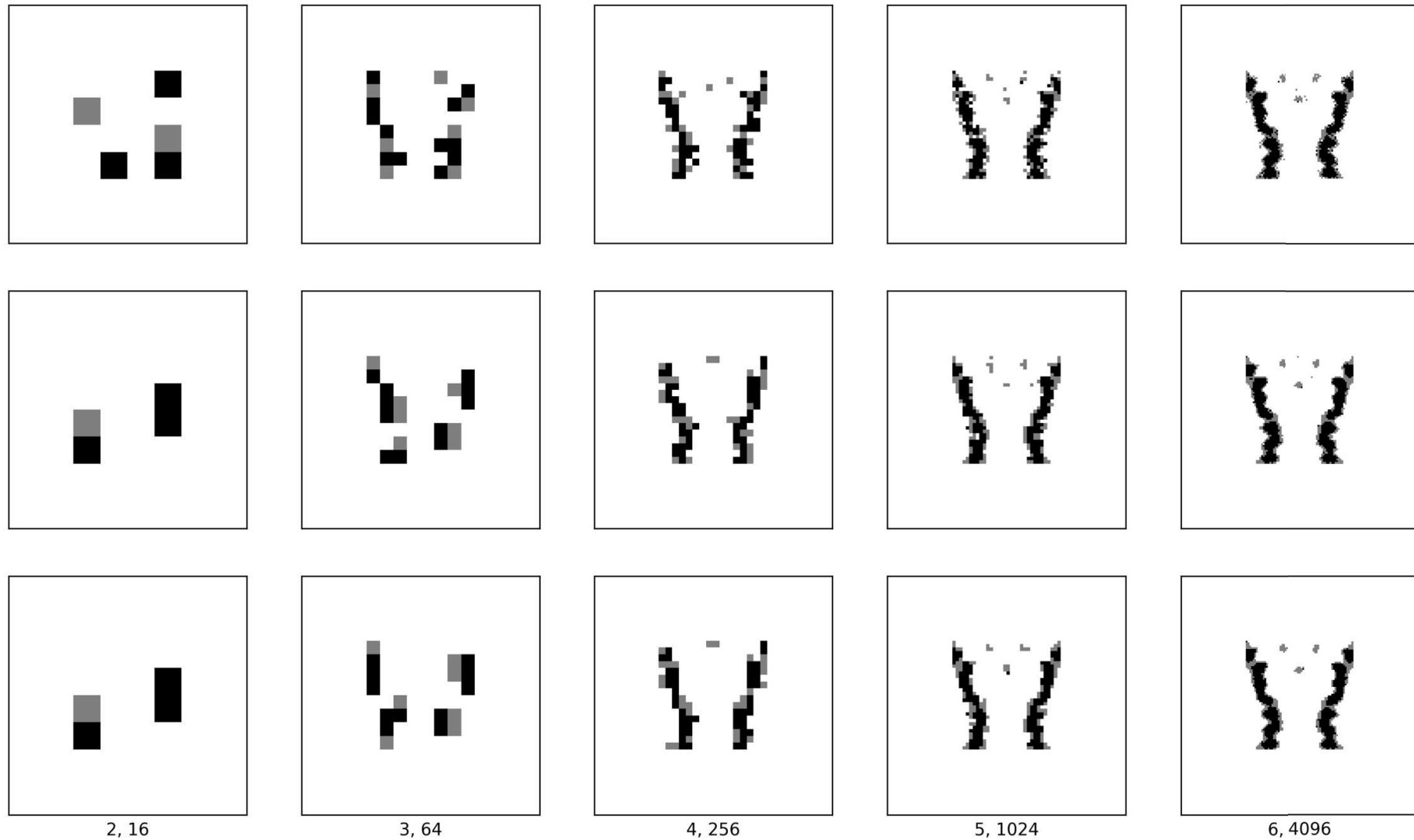
# Run Times



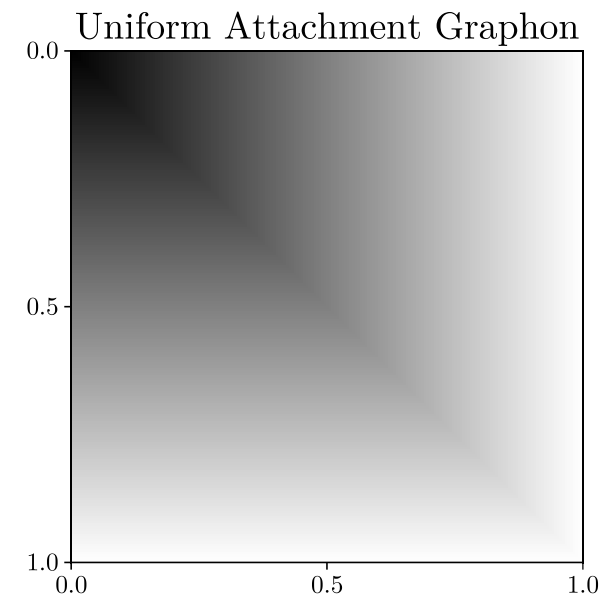
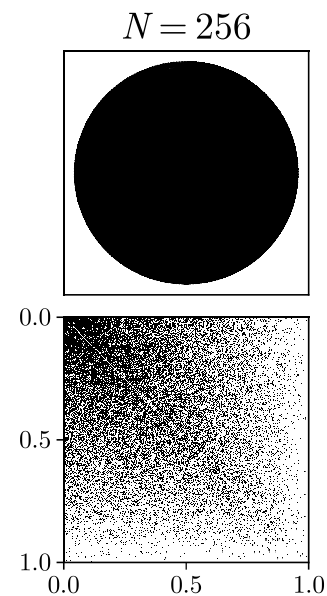
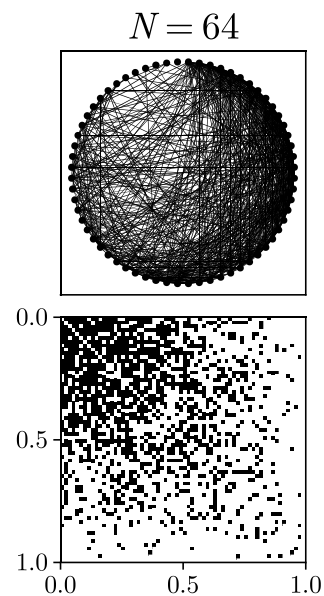
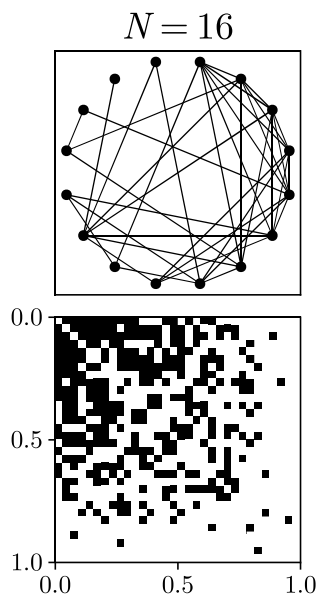
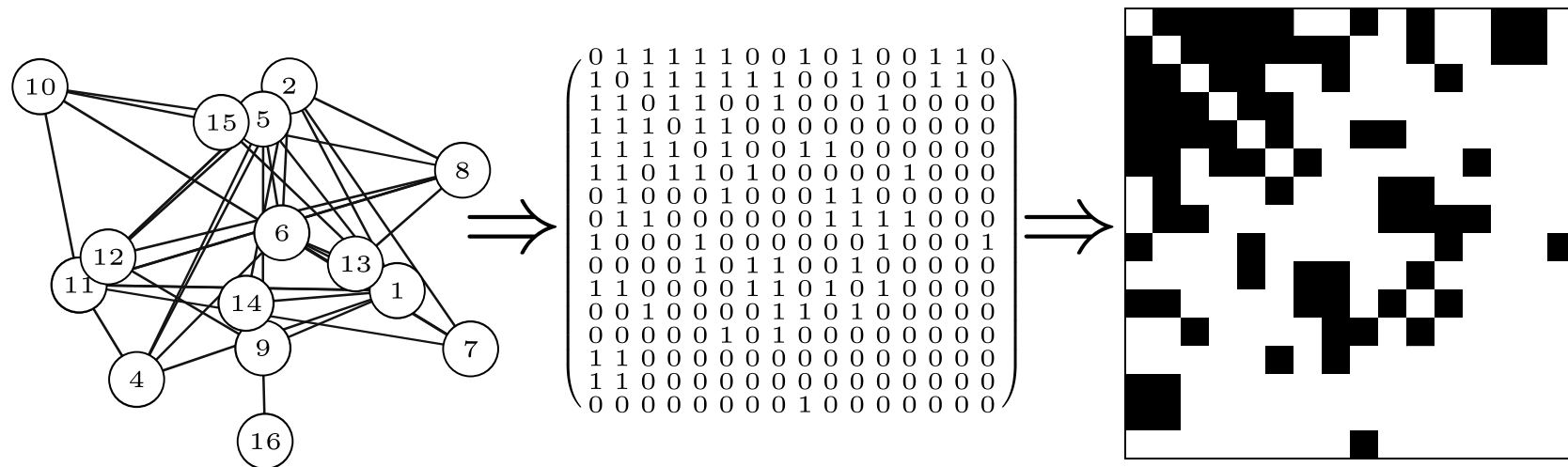
# Rounding via Shortest Paths in 2D



# Rounding via Shortest Paths in 2D



# From 2D domains to graph embeddings



# Graphon dynamical systems

M Köhler, C K, IEEE CSL 2025

For a pair  $A, B$  of graphons with weighted adjacency  $[0, 1]^2 \rightarrow [-1, 1]$ , the **cut norm** measures graph distance:

$$\|A - B\|_{\square} := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} A(x, y) - B(x, y) \, dx \, dy \right|.$$

Computing  $\|A - B\|_{\square}$  is NP-hard, but reasonable approximations exist.

A graph(on) dynamical system evolves according to

$$\dot{x} = (Af)(x) = \int_0^1 A(x, y) f(y) \, dy, \quad x \in [0, 1].$$

A graph is obtained by discretizing a graphon into  $n$  vertices; the graphon values are edge probabilities. We are interested in optimal control solutions on a **hierarchy of graphs** sampled from a **graphon**.

# Semilinear optimal graphon control

M Köhler, C K, IEEE CSL 2026

A class of semilinear optimal graphon control problems:

$$\begin{aligned} \min_{y,u} J(y,u) &:= \int_0^{t_f} \ell(t, y(t), u(t)) \, dt \\ \text{s.t. } \dot{y} &= (Ay)(t) + (Bu)(t) + f(t, y(t), u(t)) \text{ a.e } t \in [0, t_f] \\ y(0) &= y_0 \in L^2([0, 1]) \\ u(t) &\in U_{ad}(t) \subset L^2([0, 1]). \end{aligned}$$

Let  $A^n, B^n$  be  $n$ -vertex graphs randomly sampled from graphons  $A, B$ .

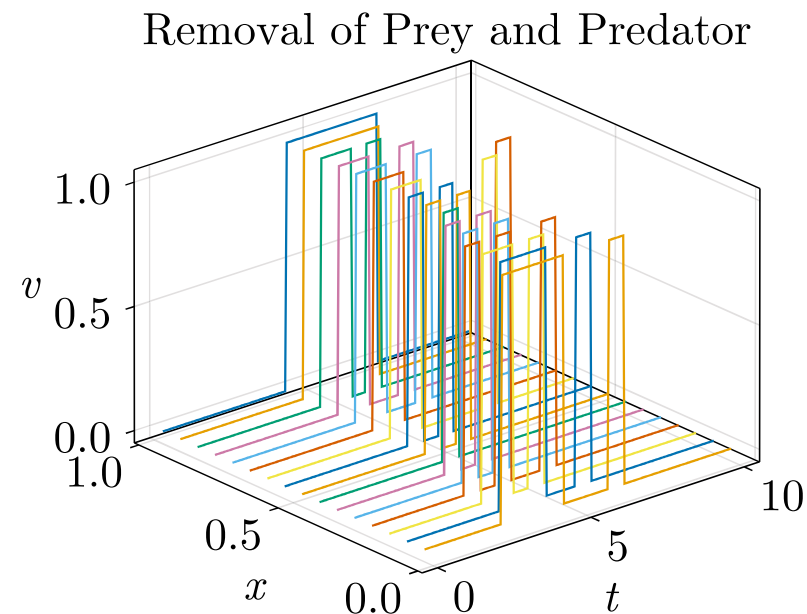
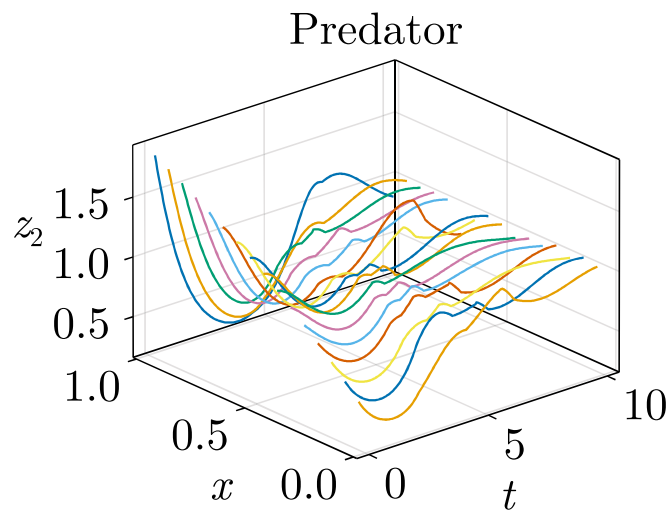
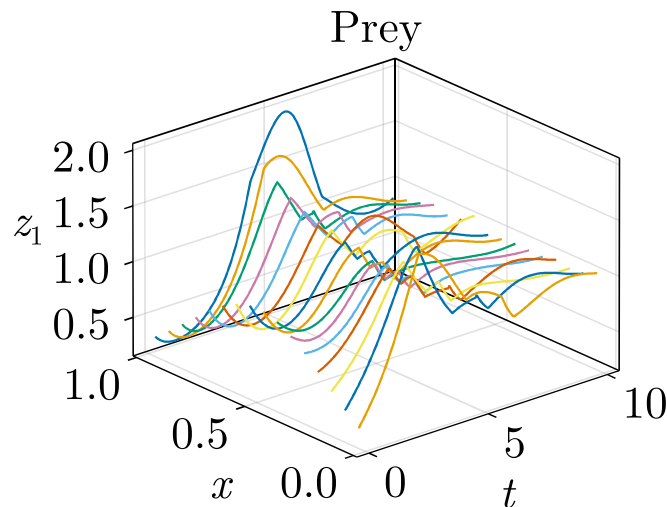
**Theorem:** Under Lipschitz and boundedness assumptions and for strictly convex and weakly lsc  $J$ , convergence

$$\|A - A^n\|_{\square} \rightarrow 0, \quad \|B - B^n\|_{\square} \rightarrow 0 \quad (n \rightarrow \infty)$$

happens at **rate**  $O(n^{-1/2})$  almost surely and induces  **$\Gamma$ -convergence** of the family semilinear finite-horizon optimal control problems associated with the hierarchy of graphs  $A^n, B^n$ .

# Optimal control of dynamics on graphons

L Schadt, C K, IEEE CSL 2026; M Köhler, C K, IEEE CSL 2026



Successful also for **other types of network dynamics**: Kuramoto, FitzHugh-Nagumo, COVID-19-SIR, ...

# Takeaway

- Mixed-Integer Optimal Control (MIOCP),  
Mixed-Integer PDE-Constrained Optimization (MIPDECO)
- Distributed integer controls, mesh dependent
- Variety of solution approaches; focus was on approximation
- Convexification and relaxation w.r.t. the  $M$ -many integer choices
- SUR induces weak-\* convergence in  $L^\infty(\Omega)$ , 1D/2D domains, graphons
- Combinatorial reconstruction MILPs below the SUR-bounds  
poly-time in  $N$ ,  $K$ , exp-time in  $M$ ; no state variables

Find references at [mo.tu-bs.de](https://mo.tu-bs.de)

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