

Optimal experimental design for spatiotemporal system identification (Part 3)

Dariusz Uciński

Institute of Control and Computation Engineering
University of Zielona Góra

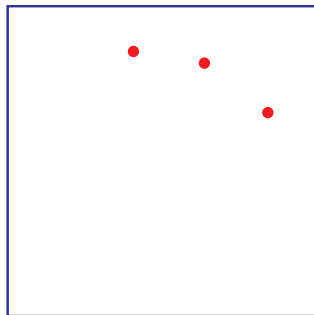
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Białystok, 19 September 2026

Mobile sensor network

Given N pointwise mobile sensors, which take the measurements in the continuous time over the interval $T = [0, t_f]$ along the observation curves $\mathbf{x}^1(\cdot), \dots, \mathbf{x}^N(\cdot)$.



0
 t_f

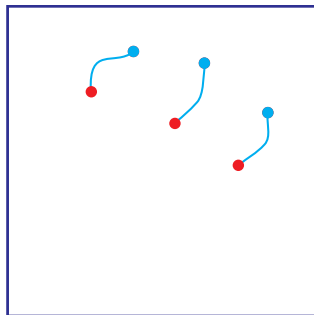


Domain Ω

• sensor
— trajectory

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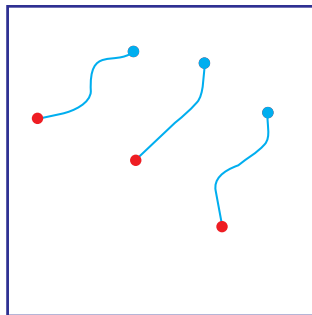
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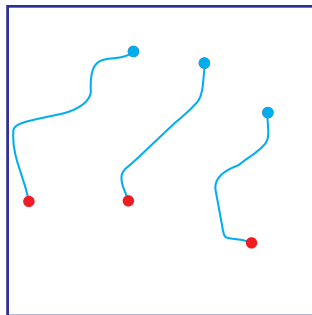


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Observations

In the case of N **mobile sensor nodes**:

Noisy measurements

$$z^j(t) = y(\mathbf{x}^j(t), t; \boldsymbol{\theta}) + \varepsilon(\mathbf{x}^j(t), t), \quad t \in T, j = 1, \dots, N$$

Notation:

- ▶ $z^j(\cdot)$ — the j -th measurement signal
- ▶ $\mathbf{x}^j(\cdot)$ — trajectory of the j -th sensor
- ▶ $\varepsilon(\cdot, \cdot)$ — space-time Gaussian white noise

$$\mathbb{E}[\varepsilon(\mathbf{x}, t)] = 0, \quad \text{var}(\varepsilon(\mathbf{x}, t)) = \sigma^2$$

LS criterion

$$\mathcal{J}(\vartheta) = \frac{1}{t_f} \sum_{j=1}^N \int_T [z^j(t) - y(\mathbf{x}^j, t; \vartheta)]^2 dt \rightarrow \min$$

Notation:

- ▶ $y(\cdot, \cdot; \vartheta)$ — the solution to the state equation corresponding to a given value of $\vartheta \in \Theta_{\text{ad}}$

Optimal measurement problem

Determine $\mathbf{x}^j(\cdot)$, $j = 1, \dots, N$ so as to maximize the identification accuracy in a sense.

LS criterion

$$\mathcal{J}(\boldsymbol{\vartheta}) = \frac{1}{t_f} \sum_{j=1}^N \int_T [z^j(t) - y(\mathbf{x}^j, t; \boldsymbol{\vartheta})]^2 dt \longrightarrow \min$$

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Optimal measurement problem

Determine $\mathbf{x}^j(\cdot)$, $j = 1, \dots, N$ so as to maximize the identification accuracy in a sense.

Question

How to quantify the identification accuracy?

Estimation accuracy

Cramér-Rao inequality

$$\text{cov } \hat{\boldsymbol{\theta}} = E\left\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right\} \succeq \mathbf{M}^{-1}$$

We have $\text{cov } \hat{\boldsymbol{\theta}} = \mathbf{M}^{-1}$ provided that an estimator is efficient! But what is $\mathbf{M}$?

Estimation accuracy

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Fisher Information Matrix (FIM)

$$\mathbf{M} = \sum_{j=1}^N \int_0^{t_f} \mathbf{g}(\mathbf{x}^j(t), t) \mathbf{g}^T(\mathbf{x}^j(t), t) dt$$

where

$$\mathbf{g}(\mathbf{x}, t) = \left(\frac{\partial y(\mathbf{x}, t; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)_{\boldsymbol{\theta}=\boldsymbol{\theta}^0}^T$$

is the vector of *sensitivity coefficients*.

Sensor dynamics

Introduce

$$\mathbf{s}(t) = (\mathbf{x}^1(t), \mathbf{x}^2(t), \dots, \mathbf{x}^N(t)), \quad \forall t \in T$$

Equations of sensor motions

$$\frac{d\mathbf{s}(t)}{dt} = \mathbf{h}(\mathbf{s}(t), \mathbf{u}(t)), \quad \mathbf{s}(0) = \mathbf{s}_0$$

Notation:

- ▷ $\mathbf{h} : \mathbb{R}^N \times \mathbb{R}^r \rightarrow \mathbb{R}^N$ — known function
- ▷ $\mathbf{s}_0 \in \Omega^N$ — the initial configuration of sensors
- ▷ $\mathbf{u} \in \mathcal{U}$ — control function
- ▷ $\mathcal{U} = \{\mathbf{u} : T \rightarrow \mathbb{R}^r \mid \mathbf{u}(\cdot) \text{ is measurable, } \mathbf{u}_{\min} \leq \mathbf{u}(t) \leq \mathbf{u}_{\max}\}$
— set of admissible controls

Optimal sensor guidance problem

Problem 1

Determine $(\mathbf{u}^*, \mathbf{s}_0^*) \in \mathcal{U} \times \Omega^N$ minimizing $J(\mathbf{u}, \mathbf{s}_0) = \Psi[\mathbf{M}(\mathbf{s}(\cdot))]$, e.g.

$$\Psi[\mathbf{M}(\mathbf{s}(\cdot))] = -\log \det \mathbf{M}(\mathbf{s}(\cdot)) \quad (\text{D-optimality})$$

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Defining

$$\mathbf{\Pi}(t) = \sum_{j=1}^N \int_0^t \mathbf{g}(\mathbf{s}(\tau), \tau) \mathbf{g}^T(\mathbf{s}(\tau), \tau) d\tau$$

we obtain a canonical control problem:

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Equivalent Mayer problem (Problem 1 reformulated)

Determine $(\mathbf{u}^*, \mathbf{s}_0^*) \in \mathcal{U} \times \Omega^N$ minimizing

$$J(\mathbf{u}, \mathbf{s}_0) = \Psi[\boldsymbol{\Pi}(t_f)]$$

subject to

$$\frac{d}{dt} \boldsymbol{\Pi}(t) = \sum_{j=1}^N \mathbf{g}(\mathbf{s}(t), t) \mathbf{g}^T(\mathbf{s}(t), t), \quad \boldsymbol{\Pi}(0) = 0$$

$$\frac{d}{dt} \mathbf{s}(t) = \mathbf{h}(\mathbf{s}(t), \mathbf{u}(t)), \quad \mathbf{s}(0) = \mathbf{s}_0$$

Computer example

Let $\Omega = (0, 1)^2$ and $T = (0, 1)$. Consider the heat equation

$$\frac{\partial y(\mathbf{x}, t)}{\partial t} = \frac{\partial}{\partial x_1} \left(\kappa(\mathbf{x}) \frac{\partial y(\mathbf{x}, t)}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\kappa(\mathbf{x}) \frac{\partial y(\mathbf{x}, t)}{\partial x_2} \right) + 20 \exp(-50(x_1 - t)^2), \quad (\mathbf{x}, t) \in \Omega \times T$$

$$y(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega$$

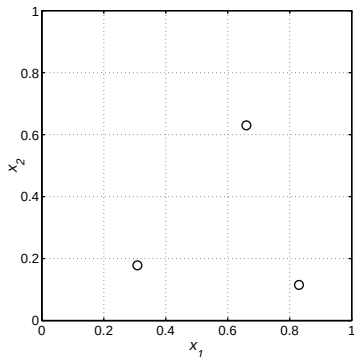
$$y(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times T$$

where $\kappa(\mathbf{x}) = \theta_1 + \theta_2 x_1 + \theta_3 x_2$,

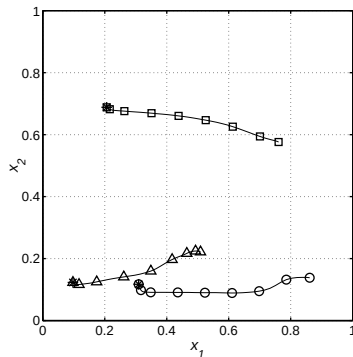
$\theta_1 = 0.1$, $\theta_2 = -0.05$, $\theta_3 = 0.2$, $\dot{s}(t) = u(t)$, $s(t_0) = s_0$,

$|u_i(t)| \leq 0.7$, $i = 1, \dots, 6$

Computer example

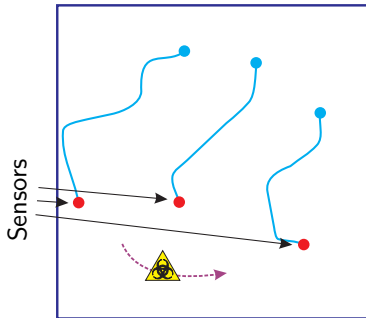


$$\det(\mathbf{M}) = 417.3$$



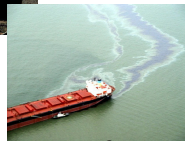
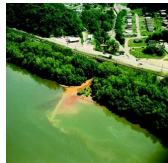
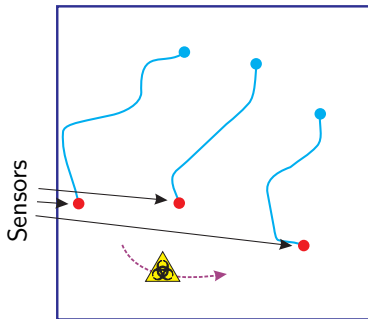
$$\det(\mathbf{M}) = 1563$$

Source identification problem



- **Problem:** How to determine the 'best' sensor trajectories for reliable source detection or/and localization?

Source identification problem



- **Problem:** How to determine the 'best' sensor trajectories for reliable source detection or/and localization?
- **Problem 2:** How to deal with **unknown** trajectories of moving source?

System description

Consider domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary Γ .

PDE model

$$\begin{aligned}\frac{\partial y}{\partial t} &= \mathcal{A}(\mathbf{x}, t, y) + f(\mathbf{x}, t), & (\mathbf{x}, t) &\in \Omega \times T \\ y(\mathbf{x}, 0) &= 0, & \mathbf{x} &\in \Omega \\ \mathcal{B}(\mathbf{x}, t, y) &= 0, & (\mathbf{x}, t) &\in \Gamma \times T\end{aligned}$$

Notation:

- $\mathbf{x}$ – spatial variable, t – time, $T = (0, t_f)$;
- $y = y(\mathbf{x}, t)$ – state vector;
- $\mathcal{A}, \mathcal{B}$ – some known operators;
- $f(\mathbf{x}, t)$ – unknown source term

Source term

Source modelling

$$f(\mathbf{x}, t) = g(\mathbf{x} - \boldsymbol{\xi}(t); \boldsymbol{\beta})w(t; \boldsymbol{\varrho})$$

- $g(\cdot; \boldsymbol{\beta})$ – spatial distribution of the source
- $\boldsymbol{\xi} : [0, t_f] \rightarrow \Omega \cup \Gamma$ – trajectory of the source
- $w(\cdot; \boldsymbol{\varrho})$ – temporal intensity of the source

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Parametrization of source trajectories

$$\boldsymbol{\xi}(t) = \boldsymbol{\eta}(t; \boldsymbol{\alpha}), \quad t \in T$$

- $\boldsymbol{\eta}(\cdot; \boldsymbol{\alpha})$ – known continuous function, e.g., splines

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$\theta = (\alpha, \beta, \varrho)$ – vector of *unknown* (constant) parameters

Challenges in algorithmic design of sensor paths

Main impediments

- dependence of optimal solutions on the true values θ_{true} of the parameters to be estimated
- off-line character of the design procedure
- source detection required before identification phase
- potential ill-posedness of the estimation procedure

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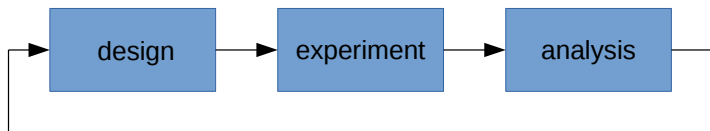
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Possible remedy

A way to obtain the on-line robust numerical procedure is to provide adaptivity through an application of a sequential design and model-predictive-control techniques.

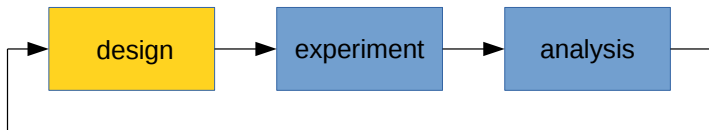
General scheme of adaptive design

Experimentation and estimation steps are alternated. At each stage, a locally optimal sensor location is determined based on the current parameter estimates (nominal parameter values can be assumed as initial guesses for the first stage), measurements are taken at the newly calculated sensor positions, and the data obtained are then analysed and used to update the parameter estimates.



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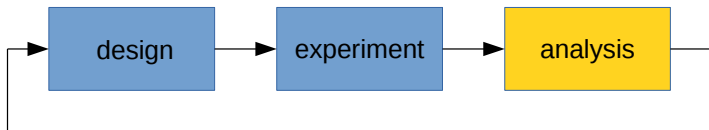
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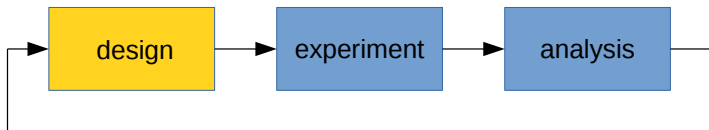
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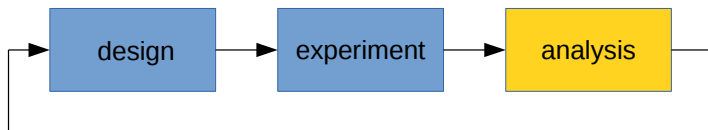
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Source detection

$$\boldsymbol{\theta}^T = [\theta_1 \dots \theta_p | \theta_{p+1} \dots \theta_d] = \left[\boldsymbol{\mu}^T \middle| \boldsymbol{\varsigma}^T \right]$$

- $\boldsymbol{\mu}$ — vector of p essential parameters in detection (localization)
- $\boldsymbol{\varsigma}$ — parameters of secondary importance

Source detection

$$\theta^T = [\theta_1 \dots \theta_p | \theta_{p+1} \dots \theta_d] = \left[\mu^T \mid \varsigma^T \right]$$

- μ — vector of p essential parameters in detection (localization)
- ς — parameters of secondary importance

Null hypothesis

$$H_0 : \mu = \mu_0$$

- μ_0 — nominal value of the vector μ corresponding to the normal system performance without a contaminant

Source detection

$$\boldsymbol{\theta}^\top = [\theta_1 \dots \theta_p | \theta_{p+1} \dots \theta_d] = \left[\boldsymbol{\mu}^\top \mid \boldsymbol{\varsigma}^\top \right]$$

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Null hypothesis

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$$

- $\boldsymbol{\mu}_0$ — nominal value of the vector $\boldsymbol{\mu}$ corresponding to the normal system performance without a contaminant

Log-likelihood ratio

Setting $\Theta_0 = \{\boldsymbol{\theta} \in \Theta_{\text{ad}} : \boldsymbol{\mu} = \boldsymbol{\mu}_0\}$, we have

$$\lambda(\mathbf{z}) = 2 \ln \frac{\sup_{\boldsymbol{\vartheta} \in \Theta_{\text{ad}}} \mathcal{L}(\mathbf{z}; \boldsymbol{\vartheta})}{\sup_{\boldsymbol{\vartheta} \in \Theta_0} \mathcal{L}(\mathbf{z}; \boldsymbol{\vartheta})} = -\frac{1}{\sigma^2} \left(\mathcal{J}(\hat{\boldsymbol{\theta}}) - \mathcal{J}(\tilde{\boldsymbol{\theta}}) \right)$$

where $\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\vartheta} \in \Theta_{\text{ad}}} \mathcal{J}(\boldsymbol{\vartheta})$, $\tilde{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\vartheta} \in \Theta_0} \mathcal{J}(\boldsymbol{\vartheta})$.

Detection rule

Theorem (Goodwin and Payne, 1977)

Under the null hypothesis H_0 , the random variable $\lambda(\mathbf{z})$ is weakly convergent to a χ^2 variable with p degrees of freedom, i.e.

$$\lambda(\mathbf{z}) \xrightarrow{\text{law}} \zeta$$

where $\zeta \sim \chi^2(p)$.

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where $\zeta \sim \chi^2(p)$.

$$S = \begin{cases} S^1 & \text{if } \lambda(\mathbf{z}) \geq k_\gamma \quad (\text{reject } H_0) \\ S^0 & \text{if } \lambda(\mathbf{z}) < k_\gamma \quad (\text{accept } H_0) \end{cases}$$

Notation:

- ▷ γ — significance level (probability of rejecting the true null hypothesis)
- ▷ k_γ — $(1 - \gamma)$ -order quantile of $\chi^2(p)$ distribution

Optimal source detection problem

Question

How to maximize the reliability of detection (i.e., minimize the probability of missed detection)?

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Answer

Maximize the power of the test for the alternative hypothesis $H_1 : \mu \neq \mu_0$ via

- increasing the number of sensor nodes N

Optimal source detection problem

Question

How to maximize the reliability of detection (i.e., minimize the probability of missed detection)?

Answer

Maximize the power of the test for the alternative hypothesis $H_1 : \mu \neq \mu_0$ via

- increasing the number of sensor nodes N
- employing the D_s -optimality criterion (Goodwin and Payne, 1977):

where

$$\Psi(M) = \det[M_{\mu\mu} - M_{\mu\varsigma} M_{\varsigma\varsigma}^{-1} M_{\mu\varsigma}^T]$$

$$M = \left[\begin{array}{c|c} \overbrace{M_{\mu\mu}}^p & \overbrace{M_{\mu\varsigma}}^{d-p} \\ \hline \overbrace{M_{\mu\varsigma}^T}^p & \overbrace{M_{\varsigma\varsigma}}^{d-p} \end{array} \right] \left. \vphantom{\begin{array}{c|c} \overbrace{M_{\mu\mu}}^p & \overbrace{M_{\mu\varsigma}}^{d-p} \\ \hline \overbrace{M_{\mu\varsigma}^T}^p & \overbrace{M_{\varsigma\varsigma}}^{d-p} \end{array}} \right\} \begin{array}{l} p \\ d-p \end{array}$$

Detection block

Step 1.

Set current design subinterval and prediction horizon:

$$T_k = (t_{k-1}, t_k], \quad \tilde{T}_k = (t_{k-1}, t_{k-1} + \Delta T_p]$$

Step 2. ANALYSIS PHASE

- 1 Parameter estimation $\Rightarrow \hat{\theta}_k$
- 2 Prediction of the system evolution on $\tilde{T}_k \Rightarrow y_{\text{pred}}$.

Step 3. DESIGN PHASE

Solve the optimal control problem given $\hat{\theta}_k$ and y_{pred} .

Step 4. EXPERIMENT PHASE

Collect observations along the designed sensor trajectories.

Step 5. DETECTION PHASE

Perform generalized log-likelihood ratio test using gathered data. If a source is detected, then STOP, otherwise GOTO Step 1.

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Simulation example

Consider the air pollutant transport-chemistry process

$$\begin{aligned}\frac{\partial y(\mathbf{x}, t)}{\partial t} + \nabla \cdot (\mathbf{v}(\mathbf{x}, t)y(\mathbf{x}, t)) \\ = \nabla \cdot (\kappa \nabla y(\mathbf{x}, t)) + f(\mathbf{x}, t), \quad t \in T = [0, 400], \quad \mathbf{x} \in \Omega = [0, 1]\end{aligned}$$

subject to the boundary and initial conditions:

$$\frac{\partial y(\mathbf{x}, t)}{\partial \mathbf{n}} = 0, \quad (\mathbf{x}, t) \in \Gamma \times T; \quad y(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega$$

- ▶ $y(\mathbf{x}, t)$ – pollutant concentration; ▶ $\mathbf{v}(\mathbf{x}, t)$ – wind velocity field,
- ▶ κ - turbulent diffusivity coefficient ($\kappa = 90 \text{ [m}^2/\text{s]}$),

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subject to the boundary and initial conditions:

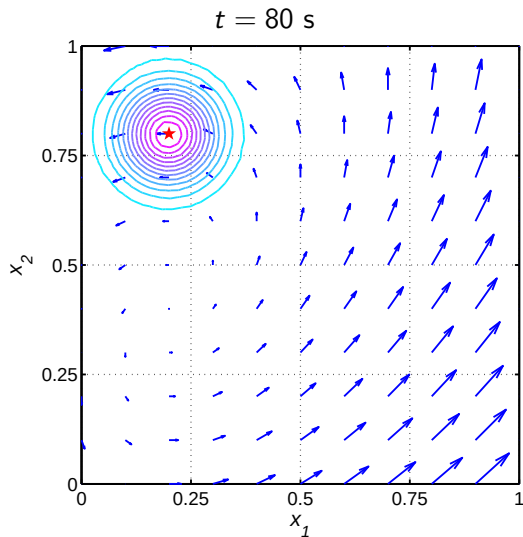
$$\frac{\partial y(\mathbf{x}, t)}{\partial \mathbf{n}} = 0, \quad (\mathbf{x}, t) \in \Gamma \times T; \quad y(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega$$

- ▶ $y(\mathbf{x}, t)$ – pollutant concentration; ▶ $\mathbf{v}(\mathbf{x}, t)$ – wind velocity field,
- ▶ κ – turbulent diffusivity coefficient ($\kappa = 90 \text{ [m}^2/\text{s]}$),

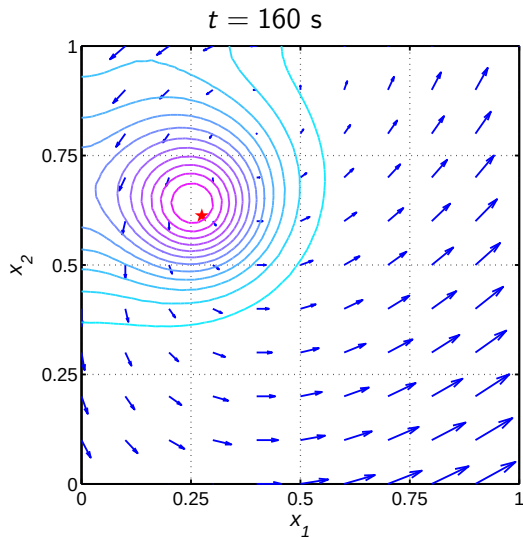
$$f(\mathbf{x}, t) = H_{t_s}(t) \varrho \exp(-100 \|\mathbf{x} - \boldsymbol{\xi}(t)\|^2)$$

- $H_{t_s}(\cdot)$ – Heaviside function ($t_s = 80 \text{ [s]}$),
- ϱ – source intensity ($\varrho = 10 \text{ [kg/s]}$),
- $\xi_{s,1} = (3t + 400)/3200 \text{ [km]}$, $\xi_{s,2} = (t - 720)^2 / 512000 \text{ [km]}$,
 $\boldsymbol{\xi}(0) = (0.2, 0.8)$, $\boldsymbol{\xi}(400) = (0.5, 0.2)$

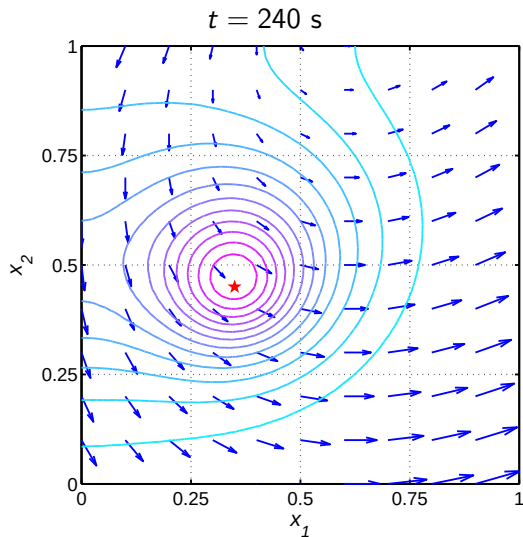
Pollutant concentration



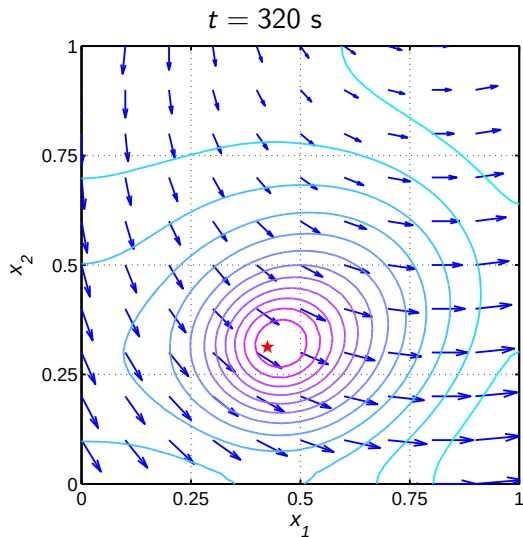
Pollutant concentration



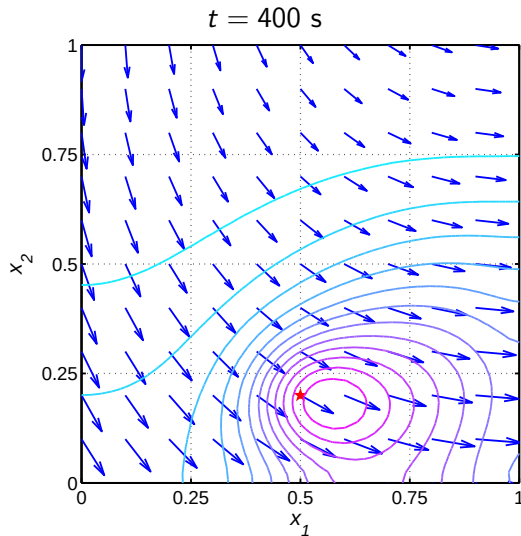
Pollutant concentration



Pollutant concentration



Pollutant concentration



Simulation setting

Purpose

Find adaptive D_s -optimal trajectories for $N = 4$ mobile sensor nodes which yield the most accurate estimates of the source intensity and position.

Simulation setting

Purpose

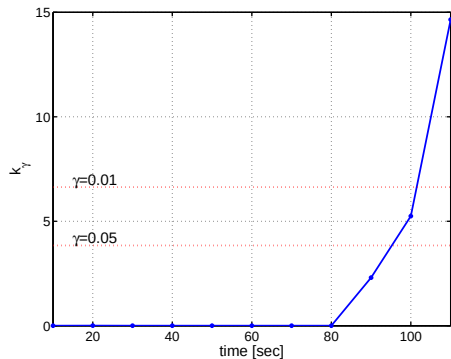
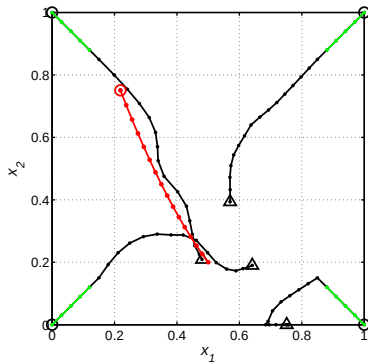
Find adaptive D_s -optimal trajectories for $N = 4$ mobile sensor nodes which yield the most accurate estimates of the source intensity and position.

- Sensors motions: $\frac{d}{dt}s(t) = u(t), \quad s(0) = s_0.$
- Control constraints: $|u_i(t)| \leq 11.0 \text{ [km/h]}.$
- Measurements are assumed to be collected at multiples of $\Delta t = 10 \text{ s}$. The observation interval was divided into 20 equal design subintervals $T_k, k = 1, \dots, 20$ (20 sec. per each time slot).
- The null hypothesis $H_0 : \varrho = 0$ is related to the zero intensity of the source.
- In the scenario the sensors are assumed to approach the contaminated area starting from the corners of the square domain towards the center with half of their maximum velocities.

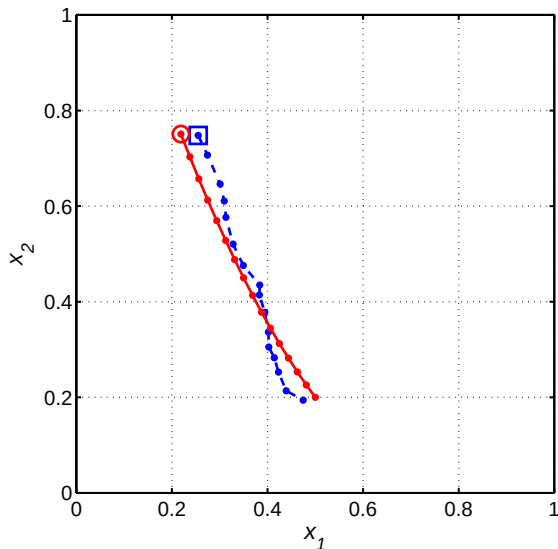
Simulation setting (ctd')

- In each subinterval T_k , a trajectory of the source $\xi(t)$ is approximated using linear splines parametrized by the current position $(\xi_{k,1}, \xi_{k,2})$ and velocity $(v_{k,1}, v_{k,2})$ of the source.
- At the k -th stage, from among the system parameters $\theta^k = (\varrho, t_s, \xi_{k,1}, \xi_{k,2}, v_{k,1}, v_{k,2}, \kappa)$, we are interested in the first four describing the current source location and intensity.
- Implementation: MATLAB based program
 - system of PDEs solved using COMSOL 3.4 and stored,
 - numerical solution of PDEs + additive measurement noise with standard deviation $\sigma = 10^{-2}$ [kg/m³] $\Rightarrow$ reference output signal,
 - resulting control problems solved using RIOTS_95 (Schwartz, 1996).
 - Multi-start procedure to avoid convergence to local optima.

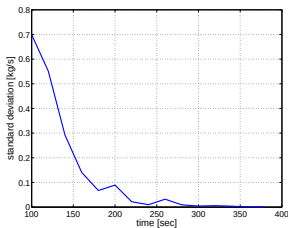
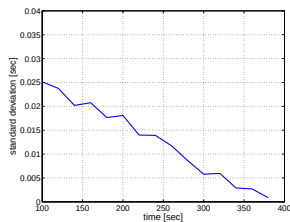
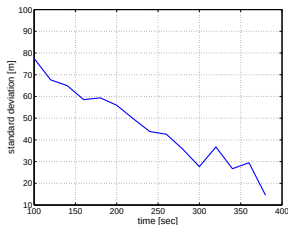
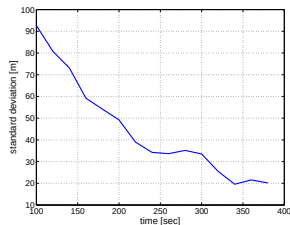
Results—Sensor trajectories



Results—Source trajectory



Results—estimates standard deviations

intensity μ starting time t_s coordinate ξ_1 coordinate ξ_2 

Concluding remarks

- An approach to moving source detection/localization in DPSs dedicated for observations using a **mobile sensor network**.
- Formulation of the optimal sensor guidance problem in terms of the Mayer problem paves the way for direct adaptation of existing control techniques and numerical solvers.
- An attempt was also made at applying a nonlinear MPC-like approach to build an adaptive on-line scheme for mobile observations.
- Open problems:
 - decomposition of the optimization task dedicated for the decentralized sensor network,
 - the control problem in the Mayer form is still hard one and requires new effective methods for large scale tasks.