

Part 1: Elementary Introduction to Fractional Calculus

Yuri Luchko

Department of Mathematics, Physics, and Chemistry
Berlin University of Applied Sciences and Technology
Berlin, Germany

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Outline

- Historical Remarks
- Preliminaries: Number Sets, Powers, Euler's Gamma Function
- Definitions and Basic Properties of Derivatives and Integrals of Non-Integer Order



Number Sets

Leopold Kronecker (German mathematician, 1823-1891):

"God made the integers, all else is man's work."

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \dots$$

Definition of Powers a^b

Let $a \in \mathbb{R}$ (base of the power a^b) and $b = n \in \mathbb{N}$ (exponent of the power a^b).

$$a^b = a^n := \begin{cases} a, & n = 1, \\ \underbrace{a \cdot \dots \cdot a}_{n\text{-times}}, & n = 2, 3, \dots \end{cases}$$

Most important property (power law):

$$a^{n_1} \cdot a^{n_2} = \underbrace{a \cdot \dots \cdot a}_{n_1\text{-times}} \cdot \underbrace{a \cdot \dots \cdot a}_{n_2\text{-times}} = \underbrace{a \cdot \dots \cdot a}_{(n_1+n_2)\text{-times}} = a^{n_1+n_2}.$$

Definition of Powers a^b

Extension to exponents $b \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots$

Requirements:

- 1) The definition of a^b agrees with the definition of a^n for $b = n \in \mathbb{N}$.
- 2) The power law $a^{b_1} \cdot a^{b_2} = a^{b_1+b_2}$ holds.

It is very important to specify the correct requirements when extending mathematical objects!

Definition of Powers a^b

Extension to exponents $b \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots$ proceeds step by step (assuming $a > 0$):

$$1) \ b = 0: a^b \cdot a^0 = a^{b+0} = a^b \Rightarrow a^0 = 1.$$

$$2) \ b = -n: a^n \cdot a^{-n} = a^{n-n} = a^0 = 1 \Rightarrow a^{-n} = \frac{1}{a^n}.$$

$$3) \ b = 1/q, \ q \in \mathbb{N}: \underbrace{a^{1/q} \cdot \dots \cdot a^{1/q}}_{q\text{-times}} = a^{q \cdot 1/q} = a^1 = a \Rightarrow a^{1/q} = \sqrt[q]{a} \text{ with}$$

the property $(\sqrt[q]{a})^q = a$.

$$4) \ b = p/q, \ p \in \mathbb{Z}, \ q \in \mathbb{N}: \underbrace{a^{p/q} \cdot \dots \cdot a^{p/q}}_{q\text{-times}} = a^{q \cdot p/q} = a^p \Rightarrow a^{p/q} = \sqrt[q]{a^p}.$$

$$5) \ b \in \mathbb{R}, \ b \notin \mathbb{Q}: \exists r_n \rightarrow b, \ r_n \in \mathbb{Q}, \ n = 1, 2, \dots \Rightarrow a^b = \lim_{n \rightarrow \infty} a^{r_n}.$$

and so on.

Another Method for Defining Powers

Idea: Derive a formula (representation) for powers with natural exponents that can be interpreted for arbitrary exponents.

Starting point: Power function $y = f(x) = (1 + x)^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

1) $\alpha = 1$:

$$(1 + x)^1 = 1 + x.$$

2) $\alpha = 2$: First binomial formula $(a + b)^2 = a^2 + 2ab + b^2 \Rightarrow$

$$(1 + x)^2 = 1 + 2x + x^2.$$

Another Method for Defining Powers

3) $\alpha = n \in \mathbb{N}$: Binomial Theorem $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \Rightarrow$

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k,$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}, \quad n! = 1 \cdot 2 \cdots n.$$

Important remark: The above formula can also be written as:

$$(1 + x)^n = \sum_{k=0}^{+\infty} \binom{n}{k} x^k,$$

because

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = 0 \quad \text{for } k \geq n+1.$$

Another Method for Defining Powers

4) General power function $y = f(x) = (1 + x)^\alpha$, $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$:

$$n \mapsto \alpha \text{ in the formula } (1 + x)^n = \sum_{k=0}^{+\infty} \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} x^k \Rightarrow$$

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2) \cdots (\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2) \cdots (\alpha-k+1)}{k!}.$$

In this formula, the general power function $(1 + x)^\alpha$ is represented using integer powers x^k , $k = 0, 1, 2, \dots$.

This formula is known in analysis as the Taylor expansion of the general power function at the point $x = 0$.

Euler's Gamma Function

In a similar way, we want to extend derivatives of integer order to arbitrary orders. For this, we need a special function: Euler's Gamma function, which generalizes the factorial function $n!$, $n \in \mathbb{N}$ to arbitrary real (complex) arguments.

Key properties of $n!$, which uniquely determine $n!$:

$$\begin{cases} 1! = 1, \\ n! = n \cdot (n-1)!, \quad n = 2, 3, \dots \end{cases}$$

Euler's Gamma Function

How can we reasonably define $\alpha!$ for $\alpha \in \mathbb{R}$, $\alpha \notin \mathbb{N}$?

Desired properties of $\alpha!$:

I. $\alpha! = n!$ if $\alpha = n \in \mathbb{N}$.

II. $\alpha! = \alpha \cdot (\alpha - 1)!$ ($n! = n \cdot (n - 1)!$).

To ensure uniqueness of $\alpha!$, some additional (very natural) requirements must be imposed.

Standard notation: $\Gamma(\alpha) = (\alpha - 1)!$ (in particular, $\Gamma(n) = (n - 1)!$, $n \in \mathbb{N}$).

Euler's Gamma Function

Starting point: the identity

$$(n-1)! = \int_0^{\infty} e^{-x} x^{n-1} dx, \quad n \in \mathbb{N}.$$

$n \mapsto \alpha$ in the above formula leads to the Euler definition of the Gamma function (Euler's integral of the second kind):

$$(\alpha-1)! = \Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \quad \alpha \in \mathbb{R}, \quad \alpha > 0.$$

Euler's Gamma Function

There are several equivalent definitions of Euler's Gamma function. All are NON-elementary!

Euler's definition (Euler's integral of the second kind):

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \quad \alpha > 0.$$

Euler-Gauss definition:

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n! n^{\alpha}}{\alpha \cdot (\alpha + 1) \cdot \dots \cdot (\alpha + n)}, \quad \alpha \neq 0, -1, -2, \dots$$

Weierstrass definition:

$$\frac{1}{\Gamma(\alpha)} = \alpha \cdot e^{\gamma \alpha} \prod_{k=1}^{\infty} \left(\left(1 + \frac{\alpha}{k} \right) e^{-\alpha/k} \right), \quad \alpha \in \mathbb{R},$$

$$\gamma := \lim_{n \rightarrow \infty} \left(\sum_{m=1}^n \frac{1}{m} - \ln(n) \right) \text{ the Euler-Mascheroni constant.}$$

Euler's Gamma Function

Properties of Euler's Gamma function:

I. $\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$

II. $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha), \quad \alpha \in \mathbb{R}.$

III. $\Gamma(\alpha+n) = \alpha \cdot (\alpha+1) \dots (\alpha+k) \cdot \Gamma(\alpha), \quad n = 2, 3, \dots$

Computation of the Gamma function for $\alpha \notin \mathbb{N}$ is not elementary!

Example: $\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi}.$

Problem Statement

Problem: Extend the sequence of operators (derivatives and integrals of integer order)

$$\dots, \frac{d^2 f}{dx^2}, \frac{df}{dx}, f(x), \int_0^x f(x_1) dx_1, \int_0^x \int_0^{x_1} f(x_2) dx_2 dx_1, \dots$$

to operators of non-integer order.

For example, define the derivative of order $1/2$:

$$\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = ???$$

Operators are mappings that assign a function to another function (functions assign a number to a number). Examples of operators are derivatives and integrals.

Properties of Derivatives and Integrals of Integer Order

Notations ($n \in \mathbb{N}$):

$$(D^n f)(x) := \frac{d^n f}{dx^n}, \quad (I^n f)(x) := \int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} f(x_n) dx_n \cdots dx_1.$$

We want to preserve the most important properties of derivatives and integrals of integer order for non-integer order operators:

I. Semigroup property (power law):

$$(I^{n_1} I^{n_2} f)(x) = (I^{n_1+n_2} f)(x), \quad n_1, n_2 \in \mathbb{N}.$$

II. First Fundamental Theorem of Calculus: $\frac{d}{dx} \left(\int_0^x f(t) dt \right) = f(x)$ or more generally $(D^n I^n f)(x) = f(x)$.

Properties of Derivatives and Integrals of Integer Order

Just as with powers, we then define (with $x^0 = 1$):

$$(I^0 f)(x) := f(x) \Leftarrow (I^n I^0 f)(x) = (I^{n+0} f)(x) = (I^n f)(x),$$

$$(D^0 f)(x) := f(x) \Leftarrow (D^0 I^0 f)(x) = (D^0 f)(x) = f(x)$$

as well as the interpretation

$$(D^n f)(x) = (I^{-n} f)(x) \Leftarrow$$

$$(D^n I^n f)(x) = f(x) = (I^0 f)(x) = (I^{n-n} f)(x) = (I^{-n} I^n f)(x).$$

Definition of D^α by Liouville

For the exponential function $y = f(x) = e^{ax}$, $a \in \mathbb{R}$ we have:

$$(D^1 f)(x) = ae^{ax}, (D^2 f)(x) = a^2 e^{ax}, \dots, (D^n f)(x) = a^n e^{ax}.$$

The derivative of order $\alpha \in \mathbb{R}$ of the exponential function $y = e^{ax}$ is then defined as:

$$n \mapsto \alpha \text{ in the formula } (D^n f)(x) = a^n e^{ax} \Rightarrow \\ (D^\alpha f)(x) = a^\alpha e^{ax}, f(x) = e^{ax}.$$

In particular:

$$(D^{\frac{1}{2}} e^{ax})(x) = \sqrt{a} e^{ax}.$$

Generalization for functions $y = f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$:

$$(D^\alpha f)(x) = \sum_{k=0}^{\infty} c_k a_k^\alpha e^{a_k x}.$$

This definition is not satisfactory!

Definition of D^α by Euler

For the power function $y = f(x) = x^a$, $a \in \mathbb{R}$ we have:

$$(D^1 f)(x) = ax^{a-1}, \quad (D^2 f)(x) = a \cdot (a-1)x^{a-2}, \quad \dots,$$

$$(D^n f)(x) = a \cdot (a-1) \cdot \dots \cdot (a-n+1)x^{a-n}.$$

In this formula, one cannot directly substitute an arbitrary $\alpha \in \mathbb{R}$ for $n \in \mathbb{N}$! First, rewrite the formula using Euler's Gamma function via the property $\Gamma(a+1) = a\Gamma(a)$:

$$a \cdot (a-1) \cdot \dots \cdot (a-n+1) = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \Rightarrow$$

$$(D^n f)(x) = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} x^{a-n}, \quad f(x) = x^a.$$

Definition of D^α by Euler

The derivative of order $\alpha \in \mathbb{R}$ of the power function $f(x) = x^a$ is then defined (the Gamma function is meaningful for arbitrary arguments):

$$n \mapsto \alpha \text{ in the formula } (D^n f)(x) = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} x^{a-n} \Rightarrow$$

$$(D^\alpha f)(x) = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} x^{a-\alpha}, \quad f(x) = x^a.$$

Definition of D^α by Euler

In particular:

$$(D^{\frac{1}{2}}x^a)(x) = \frac{\Gamma(a+1)}{\Gamma(a+1/2)}x^{a-1/2}.$$

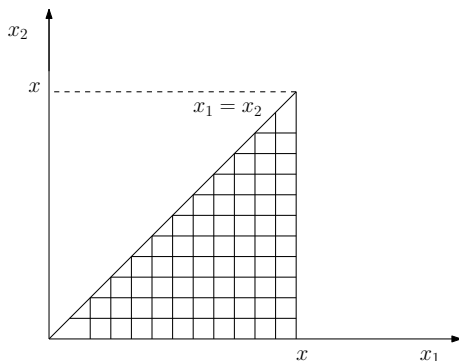
For $a = 0$, i.e., for the constant function $f(x) = x^0 = 1$:

$$(D^{\frac{1}{2}}1)(x) = \frac{\Gamma(1)}{\Gamma(1/2)}x^{-1/2} = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x}}.$$

Generalization for functions $y = f(x) = \sum_{k=0}^{\infty} c_k x^k$:

$$(D^\alpha f)(x) = \sum_{k=0}^{\infty} c_k \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}.$$

This definition is better than the first one, but still not satisfactory!

Definition of I^α by Abel (Riemann-Liouville)

$$\begin{aligned}
 (I^2 f)(x) &:= \int_0^x \int_0^{x_1} f(x_2) dx_2 dx_1 = \int_0^x \int_{x_2}^x f(x_2) dx_1 dx_2 = \\
 &\int_0^x \left(f(x_2) x_1 \Big|_{x_2}^x \right) dx_2 = \int_0^x f(x_2)(x - x_2) dx_2.
 \end{aligned}$$

Definition of I^α by Abel (Riemann-Liouville)

Analogously, one shows that the n -fold integral I^n can be represented as a single definite integral:

$$(I^n f)(x) := (I^1 I^{n-1} f)(x) = \int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} f(x_n) dx_n \cdots dx_1 =$$

$$\int_0^x f(x_n) \frac{(x - x_n)^{n-1}}{(n-1)!} dx_n = \frac{1}{\Gamma(n)} \int_0^x (x - t)^{n-1} f(t) dt.$$

The integral of order $\alpha > 0$ is then defined as:

$$n \mapsto \alpha \text{ in the formula } \frac{1}{\Gamma(n)} \int_0^x (x - t)^{n-1} f(t) dt \Rightarrow$$

$$(I^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt, \quad \alpha > 0.$$

Definition of I^α by Abel (Riemann-Liouville)

Example:

$$(I^{\frac{1}{2}}f)(x) := \frac{1}{\Gamma(1/2)} \int_0^x (x-t)^{1/2-1} f(t) dt = \\ \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt.$$

The semigroup property $((I^{n_1} I^{n_2} f)(x) = (I^{n_1+n_2} f)(x)$, $n_1, n_2 \in \mathbb{N}$) also holds for integrals of arbitrary order:

$$(I^{\alpha_1} I^{\alpha_2} f)(x) = (I^{\alpha_1+\alpha_2} f)(x), \quad \alpha_1, \alpha_2 > 0.$$

Proof using Euler's Beta function (Euler's integral of the first kind):

$$B(a, b) := \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition of D^α by Riemann-Liouville

We now want to define a **derivative** of arbitrary order $\alpha > 0$ such that the First Fundamental Theorem of Calculus $((D^n I^n f)(x) = f(x), n \in \mathbb{N})$ remains valid, i.e.,

$$(D^\alpha I^\alpha f)(x) = f(x), \alpha > 0.$$

For $\alpha > 0$, choose $n \in \mathbb{N}$ such that $n - 1 < \alpha \leq n$. Then:

$$(D^n I^{n-\alpha} I^\alpha f)(x) = (D^n I^{n-\alpha+\alpha} f)(x) = (D^n I^n f)(x) = f(x).$$

This leads to the following definition of the derivative of order $\alpha \in]n - 1, n]$:

$$(D^\alpha f)(x) := (D^n I^{n-\alpha} f)(x) = \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t) dt \right).$$

Definition of D^α by Riemann-Liouville

Example: $\alpha = 1/2$. Then $0 < \alpha = 1/2 < 1$ and $n = 1$.

The derivative of order $1/2$ is thus defined as:

$$\frac{d^{1/2} f}{dx^{1/2}} = (D^{1/2} f)(x) := (D^1 I^{1/2} f)(x) = \frac{d}{dx} \left(\frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt \right).$$

Other Definitions of D^α

The Caputo fractional derivative (introduced by Abel!) of order α ($0 < \alpha < 1$):

$$(D_C^\alpha x)(t) := (I^{1-\alpha} x')(t).$$

The generalized Riemann-Liouville or the Hilfer fractional derivative of order α ($0 < \alpha < 1$) and type γ ($0 \leq \gamma \leq 1 - \alpha$):

$$(D_H^{\alpha,\gamma} x)(t) := (I^\gamma \frac{d}{dt} I^{1-\alpha-\gamma} x)(t)$$

and many others.

1st Fundamental Theorem of FC

On the appropriate non-trivial spaces of functions, all of these fractional derivatives are left-inverse operators to the Riemann-Liouville fractional integral, i.e., they satisfy the 1st Fundamental Theorem of Fractional Calculus:

$$(D_{RL}^{\alpha} I^{\alpha} x)(t) = x(t), \quad (D_C^{\alpha} I^{\alpha} x)(t) = x(t), \quad (D_H^{\alpha, \gamma_1} I^{\alpha} x)(t) = x(t).$$

Luchko, Yu: Fractional derivatives and the fundamental theorem of Fractional Calculus. Fract. Calc. Appl. Anal. 23 (2020), 939-966.

Part 2: Generic fractional derivative and its properties

Yuri Luchko

Department of Mathematics, Physics, and Chemistry
Berlin University of Applied Sciences and Technology
Berlin, Germany

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Motivation for a generic fractional derivative

- Several unequal definitions of the fractional derivatives even in the case of functions of a single variable
- Interrelations between different kinds of the fractional derivatives are still not completely clarified
- Repeating of derivations of the same results for each definition of the fractional derivatives
- Problems in choosing an appropriate definition for the mathematical models involving fractional derivatives
- Partial solution to these problems in form of a generic fractional derivative

Outline of the talk

- Axioms for uniqueness of the fractional integral (the Riemann-Liouville fractional integral)
- Generic fractional derivative as a set of all left-inverse operators to the Riemann-Liouville fractional integral
- Basic and advanced properties of the generic fractional derivative
- Fractional relaxation equation with the generic fractional derivative

The fractional integral

Cartwright, D.I.; McMullen, J.R. A note on the fractional calculus. *Proc. of the Edinburgh Mathematical Society* **1978**, *21*, 79–80.

Restrictions:

- 1) FC operators acting on functions of a single variable,
- 2) FC operators depending on a single parameter (order of the fractional derivatives and integrals).

The fractional integral

Let X be either $L_p(0, 1)$, $1 \leq p < +\infty$ or $C([0, 1])$ and the following axioms hold true for a one-parametric family of operators I^α , $\alpha > 0$:

$$\text{A1. } (I^1 x)(t) = \int_0^t x(\tau) d\tau, \quad x \in X.$$

$$\text{A2. } (I^\alpha I^\beta x)(t) = (I^{\alpha+\beta} x)(t), \quad \alpha, \beta > 0, \quad x \in X.$$

A3. $\alpha \rightarrow I^\alpha$ is a continuous map of $(0, +\infty)$ into $\mathcal{L}(X)$ for some Hausdorff topology on $\mathcal{L}(X)$, weaker than the norm topology, where $\mathcal{L}(X)$ is the set of all linear operators acting from X to X .

A4. $x \in C([0, 1])$ and $x(t) \geq 0$ or $x \in L_p(0, 1)$ and $x(t) \geq 0$ almost everywhere on $[0, 1] \Rightarrow (I^\alpha x)(t) \geq 0$ or $(I^\alpha x)(t) \geq 0$ almost everywhere on $[0, 1]$, respectively, for all $\alpha > 0$.

The fractional integral

Cartwright, D.I.; McMullen, J.R. A note on the fractional calculus. *Proc. of the Edinburgh Mathematical Society* **1978**, 21, 79–80.

The unique family of operators that satisfy the axioms A1-A4 is the Riemann-Liouville fractional integrals

$$(I^\alpha x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} x(\tau) d\tau, \alpha > 0.$$

Thus, in a certain sense, the only true one-parameter fractional integrals I^α , $\alpha > 0$ of functions of a single variable defined on a finite interval are the Riemann-Liouville fractional integrals.

Fractional derivatives associated with the Riemann-Liouville fractional integral

The Riemann-Liouville fractional derivative of order α ($0 < \alpha < 1$):

$$(D_{RL}^{\alpha} x)(t) := \frac{d}{dt} (I^{1-\alpha} x)(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x(\tau) d\tau.$$

The Caputo fractional derivative of order α ($0 < \alpha < 1$):

$$(D_C^{\alpha} x)(t) := (I^{1-\alpha} x')(t).$$

The generalized Riemann-Liouville or the Hilfer fractional derivative of order α ($0 < \alpha < 1$) and type γ_1 ($0 \leq \gamma_1 \leq 1 - \alpha$):

$$(D_H^{\alpha, \gamma_1} x)(t) = (I^{\gamma_1} \frac{d}{dt} I^{1-\alpha-\gamma_1} x)(t)$$

and many others.

Fractional derivatives associated with the Riemann-Liouville fractional integral

On the appropriate non-trivial spaces of functions, all of these fractional derivatives are left-inverse operators to the Riemann-Liouville fractional integral, i.e., they satisfy the 1st Fundamental Theorem of Fractional Calculus:

$$(D_{RL}^{\alpha} I^{\alpha} x)(t) = x(t), \quad (D_C^{\alpha} I^{\alpha} x)(t) = x(t), \quad (D_H^{\alpha, \gamma_1} I^{\alpha} x)(t) = x(t).$$

Question: Can we somehow unify different kinds of the fractional derivatives (the Riemann-Liouville fractional derivative, Caputo fractional derivative, Hilfer fractional derivative, and many others)?

Luchko, Yu: On a Generic Fractional Derivative Associated with the Riemann-Liouville Fractional Integral. Axioms 2024, 13(9), 604.

Generic fractional derivative

Przeworska-Rolewicz, D. Concerning left-invertible operators. *Bull. de l'Acad. Pol. Sci.* **1972**, 20, 837–839.

Definition (Przeworska-Rolewicz)

An operator $R \in \mathcal{L}(X)$ with $D_R = X$ is said to be left-invertible if there exists an operator $L \in \mathcal{L}(X)$ with $D_L \subset R(X)$ such that

$$LR = E \text{ on } X,$$

where E denotes the identity operator.

The Riemann-Liouville fractional integral is left-invertible!

Generic fractional derivative

Definition

Let X be a linear space over $\mathbb{R}$ or $\mathbb{C}$ and $D_I^\alpha = X$, where I^α , $\alpha > 0$ is the Riemann-Liouville fractional integral.

The set of all linear operators $\mathbb{D}^\alpha$, $\alpha > 0$, left-inverse to the Riemann-Liouville fractional integral I^α is called a generic fractional derivative associated with the Riemann-Liouville fractional integral.

Remark

The definition postulates the 1st Fundamental Theorem of (Fractional) Calculus for any representative of the generic fractional derivative:

$$\mathbb{D}^\alpha I^\alpha = E,$$

where E denotes the identity operator.

Representatives of the generic fractional derivative

The Riemann-Liouville fractional derivative of order α ($0 < \alpha < 1$):

$$(D_{RL}^{\alpha} x)(t) := \frac{d}{dt} (I^{1-\alpha} x)(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x(\tau) d\tau.$$

The Caputo fractional derivative of order α ($0 < \alpha < 1$):

$$(D_C^{\alpha} x)(t) := (I^{1-\alpha} x')(t).$$

The generalized Riemann-Liouville or the Hilfer fractional derivative of order α ($0 < \alpha < 1$) and type γ_1 ($0 \leq \gamma_1 \leq 1 - \alpha$):

$$(D_H^{\alpha, \gamma_1} x)(t) = (I^{\gamma_1} \frac{d}{dt} I^{1-\alpha-\gamma_1} x)(t).$$

Representatives of the generic fractional derivative

The Djrbashian-Nersessian fractional derivative of order α ($0 < \alpha < 1$) and type $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$:

$$(D_{nL}^{\alpha,(\gamma)} x)(t) := \left(\prod_{k=1}^n (I^{\gamma_k} \frac{d}{dt}) \right) (I^{n-\alpha-s_n} x)(t),$$

$$s_k := \sum_{i=1}^k \gamma_i, \quad k = 1, 2, \dots, n,$$

$$0 \leq \gamma_k \quad \text{and} \quad \alpha + s_k \leq k, \quad k = 1, 2, \dots, n.$$

Projector operator of the generic fractional derivative

Definition

Let $\mathbb{D}^\alpha$ be a fractional derivative associated with the Riemann-Liouville fractional integral I^α .

The operator

$$\mathbb{P}^\alpha = E - I^\alpha \mathbb{D}^\alpha \text{ on } D_{\mathbb{D}^\alpha}$$

is called a projector operator of the fractional derivative $\mathbb{D}^\alpha$.

Remark

This formula can be rewritten in the form of the so-called 2nd Fundamental Theorem of FC for the generic fractional derivative $\mathbb{D}^\alpha$:

$$(I^\alpha \mathbb{D}^\alpha x)(t) = x(t) - (\mathbb{P}^\alpha x)(t), \quad x \in D_{\mathbb{D}^\alpha}.$$

As soon as we have an explicit formula for the projector operator $\mathbb{P}^\alpha$ of a representative of $\mathbb{D}^\alpha$, we immediately get an explicit form of the second fundamental theorem of FC for this fractional derivative.

Projector operator of the generic fractional derivative

For convenience, in what follows, we use the notation

$$h_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0. \quad (1)$$

Projector operators of the Riemann-Liouville, Caputo, Hilfer, and Djrbashian-Nersessian fractional derivatives:

$$(\mathbb{P}_{RL}^{\alpha} x)(t) := (E - I^{\alpha} D_{RL}^{\alpha} x)(t) = (I^{1-\alpha} x)(0) h_{\alpha}(t),$$

$$(\mathbb{P}_C^{\alpha} x)(t) := (E - I^{\alpha} D_C^{\alpha} x)(t) = x(0) h_1(t),$$

$$(\mathbb{P}_H^{\alpha} x)(t) := (E - I^{\alpha} D_H^{\alpha, \gamma_1} x)(t) = (I^{1-\alpha-\gamma_1} x)(0) h_{\alpha+\gamma_1}(t),$$

$$(P_{nL}^{\alpha} x)(t) := (E - I^{\alpha} D_{nL}^{\alpha, (\gamma)} x)(t) = \sum_{k=1}^n p_k h_{\alpha+s_k-k+1}(t),$$

$$s_k = \sum_{i=1}^k \gamma_i, \quad p_k = \left(\prod_{i=k+1}^n \left(I^{\gamma_i} \frac{d}{dt} \right) I^{n-\alpha-s_n} x \right) (0), \quad k = 1, 2, \dots, n.$$

Basic properties of the generic fractional derivative

Theorem

Let $\mathbb{D}^\alpha$ be a fractional derivative associated with the Riemann-Liouville fractional integral I^α defined on the linear space X , and let $\mathbb{P}^\alpha$ be its projector operator. Then the following properties hold true:

P1) The operator $\mathbb{P}^\alpha$ is a projector:

$$((\mathbb{P}^\alpha)^2 x)(t) = (\mathbb{P}^\alpha x)(t), \quad x \in D_{\mathbb{D}^\alpha}. \quad (2)$$

P2) The image of $\mathbb{P}^\alpha$ belongs to the null-space of $\mathbb{D}^\alpha$:

$$(\mathbb{D}^\alpha \mathbb{P}^\alpha x)(t) = 0, \quad x \in D_{\mathbb{D}^\alpha}. \quad (3)$$

P3) The image of I^α belongs to the null-space of $\mathbb{P}^\alpha$:

$$(\mathbb{P}^\alpha I^\alpha x)(t) = 0, \quad x \in X. \quad (4)$$

Basic properties of the generic fractional derivative

Theorem

P4) The null-space of $\mathbb{D}^\alpha$ is characterized in terms of its projector operator $\mathbb{P}^\alpha$ as follows:

$$N_{\mathbb{D}^\alpha} = \{x \in D_{\mathbb{D}^\alpha} : (\mathbb{P}^\alpha x)(t) = x(t)\}. \quad (5)$$

P5) The null-space of $\mathbb{P}^\alpha$ is characterized as follows:

$$N_{\mathbb{P}^\alpha} = \{x : x(t) = (I^\alpha \phi)(t), \phi \in X\}. \quad (6)$$

For further basic properties and examples see the recent paper

Luchko, Yu: On a Generic Fractional Derivative Associated with the Riemann–Liouville Fractional Integral. *Axioms* 2024, 13(9), 604.

Advanced properties of the generic fractional derivative

Theorem

Let $\mathbb{D}_1^\alpha$ and $\mathbb{D}_2^\alpha$ be two fractional derivatives associated with the Riemann-Liouville fractional integral, the null-space of $\mathbb{D}_1^\alpha$ be finite-dimensional ($N_{\mathbb{D}_1^\alpha} = \{c_1\phi_1(t) + \dots + c_n\phi_n(t), c_1, \dots, c_n \in \mathbb{R}\}$), where the functions $\phi_1, \dots, \phi_n$ are linearly independent, and the inclusion $N_{\mathbb{D}_1^\alpha} \subset D_{\mathbb{D}_2^\alpha}$ hold true.

Then the fractional derivatives $\mathbb{D}_1^\alpha$ and $\mathbb{D}_2^\alpha$ are connected by the relation

$$(\mathbb{D}_2^\alpha x)(t) = (\mathbb{D}_1^\alpha x)(t) + \sum_{j=1}^n a_j (\mathbb{D}_2^\alpha \phi_j)(t), \quad x \in D_{\mathbb{D}_1^\alpha},$$

where the coefficients a_j , $j = 1, \dots, n$ are determined by the relation

$$(\mathbb{P}_1^\alpha x)(t) = \sum_{j=1}^n a_j \phi_j(t).$$

Advanced properties of the generic fractional derivative

Relation between the Riemann-Liouville and Caputo fractional derivatives:

$$(D_{RL}^{\alpha} x)(t) = (D_C^{\alpha} x)(t) + x(0)h_{1-\alpha}(t), \quad x \in D_{D_C^{\alpha}}.$$

Relation between the Riemann-Liouville and the Djrbashian-Nersessian fractional derivatives:

$$(D_{RL}^{\alpha} x)(t) = (D_{nL}^{\alpha,(\gamma)} x)(t) + \sum_{k=1}^n p_k h_{s_k-k+1}(t), \quad x \in D_{D_{nL}^{\alpha,(\gamma)}},$$

$$s_k = \sum_{i=1}^k \gamma_i, \quad p_k = \left(\prod_{i=k+1}^n \left(I^{\gamma_i} \frac{d}{dt} \right) I^{n-\alpha-s_n} x \right) (0), \quad k = 1, 2, \dots, n.$$

Advanced properties of the generic fractional derivative

Theorem

Let $\mathbb{D}^\alpha$ be any fractional derivative associated with the Riemann-Liouville fractional integral I^α and $x \in D(\mathbb{D}^\alpha)^n$, $n \in \mathbb{N}$.

Then the generalized Taylor formula

$$x(t) = \sum_{k=0}^{n-1} (I^{\alpha k} (\mathbb{P}^\alpha((\mathbb{D}^\alpha)^k x)))(t) + r_n(t), \quad n \in \mathbb{N}$$

holds valid, where the remainder r_n is given by the formula

$$r_n(t) = (I^{\alpha n}((\mathbb{D}^\alpha)^n x))(t).$$

Under the condition $(\mathbb{D}^\alpha)^n x \in C([0, a])$, $a > 0$, the remainder r_n can be represented as follows:

$$r_n(t) = ((\mathbb{D}^\alpha)^n x)(c) h_{\alpha n+1}(t), \quad t \in (0, a), \quad c \in [0, t].$$

Advanced properties of the generic fractional derivative

For the Riemann-Liouville fractional derivative, the generalized Taylor formula takes the form

$$x(t) = \sum_{k=0}^{n-1} (I^{1-\alpha} (D_{RL}^{\alpha})^k x)(0) h_{\alpha k + \alpha}(t) + r_n(t),$$

$$r_n(t) = (I^{\alpha n} ((D_{RL}^{\alpha})^n x))(t), \quad n \in \mathbb{N}.$$

If the inclusion $(D_{RL}^{\alpha})^n x \in C([0, a])$, $a > 0$ holds true, the remainder r_n can be represented as follows:

$$r_n(t) = ((D_{RL}^{\alpha})^n x)(c) h_{\alpha n + 1}(t), \quad c \in [0, t].$$

Advanced properties of the generic fractional derivative

The generalized Taylor formula for the Caputo fractional derivative:

$$x(t) = \sum_{k=0}^{n-1} ((D_C^\alpha)^k x)(0) h_{\alpha k+1}(t) + r_n(t), \quad r_n(t) = (I^{\alpha n}((D_C^\alpha)^n x))(t), \quad n \in \mathbb{N}.$$

For the functions satisfying the condition $(D_C^\alpha)^n x \in C([0, a])$, $a > 0$, the remainder r_n can be represented in the Lagrange form:

$$r_n(t) = ((D_C^\alpha)^n x)(c) h_{\alpha n+1}(t), \quad c \in [0, t].$$

Advanced properties of the generic fractional derivative

The generalized Taylor formula for the Hilfer fractional derivative takes the following form:

$$x(t) = \sum_{k=0}^{n-1} (I^{1-\alpha-\gamma_1} (D_H^{\alpha,\gamma_1})^k x)(0) h_{\alpha k + \alpha + \gamma_1}(t) + r_n(t),$$

$$r_n(t) = (I^{\alpha n} ((D_H^{\alpha,\gamma_1})^n x))(t), \quad n \in \mathbb{N}.$$

Under the condition $(D_H^{\alpha,\gamma_1})^n x \in C([0, a])$, $a > 0$, the remainder r_n can be represented in the Lagrange form as follows:

$$r_n(t) = ((D_H^{\alpha,\gamma_1})^n x)(c) h_{\alpha n + 1}(t), \quad c \in [0, t].$$

Fractional relaxation equation with the generic fractional derivative

The initial-value problem for the fractional relaxation equation

$$\begin{cases} (\mathbb{D}^\alpha y)(t) = -\lambda y(t), & 0 < \alpha < 1, \lambda > 0, t > 0, \\ (\mathbb{P}^\alpha y)(t) = y_0(t), & y_0(t) \in N_{\mathbb{D}^\alpha}, \end{cases}$$

where $\mathbb{P}^\alpha$ is the projector operator of $\mathbb{D}^\alpha$, possesses a unique solution given by the formula

$$y(t) = y_0(t) - \lambda(\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda\tau^\alpha) * y_0(\tau))(t),$$

where $E_{\alpha,\beta}$ is the two-parameters Mittag-Leffler function defined by the convergent series

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \beta \in \mathbb{C}, \alpha > 0.$$

Fractional relaxation equation with the generic fractional derivative

Particular cases:

1) The initial-value problem with the Caputo fractional derivative

$$\begin{cases} (D_C^\alpha y)(t) = -\lambda y(t), & 0 < \alpha < 1, \lambda > 0, t > 0, \\ y(0) = y_1. \end{cases}$$

has the solution

$$y(t) = y_1 E_{\alpha,1}(-\lambda t^\alpha).$$

2) The initial-value problem with the Riemann-Liouville fractional derivative

$$\begin{cases} (D_{RL}^\alpha y)(t) = -\lambda y(t), & 0 < \alpha < 1, \lambda > 0, t > 0, \\ (I^{1-\alpha} y)(0) = y_1. \end{cases}$$

has the solution

$$y(t) = y_1 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha).$$

Fractional relaxation equation with the generic fractional derivative

Particular cases:

3) The initial-value problem with the Hilfer fractional derivative

$$\begin{cases} (D_H^{\alpha, \gamma_1} y)(t) = -\lambda y(t), & 0 < \alpha < 1, \lambda > 0, t > 0, \\ (I^{1-\alpha-\gamma_1} y)(0) = y_1. \end{cases}$$

has the solution

$$y(t) = y_1 t^{\alpha+\gamma_1-1} E_{\alpha, \alpha+\gamma_1}(-\lambda t^\alpha).$$

Open problems

- Do some realisations of the generic fractional derivative with a finite-dimensional null-space that are different from the Djrbashian-Nersessian fractional derivatives exist?
- Do some fractional derivatives associated with the Riemann-Liouville integrals whose null-space is infinite dimensional exist?
- Derivation of other advanced properties of the generic fractional derivative that are independent on its realisations and are based just on its definition as the set of the linear operators that are left-inverse to the Riemann-Liouville fractional integral.

Part 3: Analogies of the Fermat theorem for the general fractional derivatives

Yuri Luchko

Department of Mathematics, Physics, and Chemistry
Berlin University of Applied Sciences and Technology
Berlin, Germany

Workshop on Optimization Methods and Applications
Bialystok University of Technology
Bialystok, Poland
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Outline of the talk:

- History of origins of the general fractional derivatives (results by Abel and Sonin)
- Definitions and basic properties of the general fractional integrals and derivatives
- Analogies of the Fermat theorem for the general fractional derivatives

Tautochrone problem

In the papers

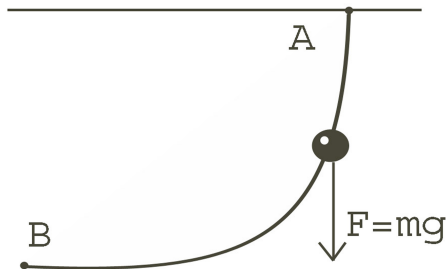
N.H. Abel: Opløsning af et par opgaver ved hjælp af bestemte integraler.
Mag. Naturvidenskaberne **1823**, 2, 2.

N.H. Abel: Auflösung einer mechanischen Aufgabe. *J. Die Reine Angew. Math.* **1826**, 1, 153–157.

Abel (05.08.1802 - 06.04.1829) considered and solved the so-called generalised tautochrone problem (from Greek prefix tauto- meaning same and chrono time).

Tautochrone problem

The tautochrone is a curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point on the curve.



It turned out that the tautochrone is a suitable part of the cycloid. This result was derived in the works of **Christiaan Huygens** (14.04.1629 - 08.07.1695) by some advanced **geometrical methods**.

Abel's model and its generalizations

$$T = \int_0^h \frac{d\phi(s)}{\sqrt{h-s}}, \quad h > 0.$$

In his papers, Abel solved a more general problem (by means of the **analytical methods**): For a given function $T = T(h)$, find a curve for which the sliding time taken by an object sliding without friction depends on the position h of its starting point on the curve as $T = T(h)$:

$$T(h) = \int_0^h \frac{d\phi(s)}{\sqrt{h-s}}, \quad h > 0$$

as well as its mathematical generalization:

$$T(h) = \int_0^h \frac{d\phi(s)}{(h-s)^\alpha}, \quad 0 < \alpha < 1.$$

Solution to the Abel equation

Under the condition $\phi(0) = 0$, the integral (in fact, **integro-differential**) equation

$$T(h) = \int_0^h (h-s)^{-\alpha} \phi'(s) ds, \quad 0 < \alpha < 1,$$

has the solution

$$\phi(h) = \frac{\sin(\alpha\pi)}{\pi} \int_0^h (h-s)^{\alpha-1} T(s) ds, \quad t > 0.$$

To derive the solution formula, Abel employed the following relation (that is valid because of the known formula for the Euler Beta-function in terms of the Gamma-funktion):

$$(h_{1-\alpha} * h_{\alpha})(t) = 1, \quad t > 0, \quad h_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad (f * g)(t) = \int_0^t f(t-\tau)g(\tau) d\tau.$$

Sonin condition

In the paper

N. Sonine: Sur la généralisation d'une formule d'Abel. *Acta Math.* **1884**, 4, 171–176

Sonin introduced a pair of functions κ , k (Sonin kernels) that satisfy the relation (Sonin condition)

$$(\kappa * k)(t) = 1, \quad t > 0$$

as a generalization of the formula used by Abel:

$$(h_\alpha * h_{1-\alpha})(t) = 1, \quad t > 0, \quad h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0.$$

Sonin integral equation

Let the kernels κ , k satisfy the Sonin condition.

Following Abel's solution method, Sonin **formally** showed that the integral equation (its right-hand side is nowadays referred to as the general fractional integral, GFI)

$$f(t) = (\mathbb{I}_{(\kappa)} \phi)(t) := (\kappa * \phi)(t) = \int_0^t \kappa(t - \tau) \phi(\tau) d\tau$$

has a solution in the following form (its right-hand side is nowadays referred to as the general fractional derivative, GFD)

$$\phi(t) = (\mathbb{D}_{(k)} f)(t) := \frac{d}{dt}(k * f)(t) = \frac{d}{dt}(\mathbb{I}_{(k)} f)(t).$$

Examples of the Sonin kernels

$$1) \kappa(t) = h_{\alpha}(t), \quad k(t) = h_{1-\alpha}(t),$$

$$2) \kappa(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}), \quad k(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}), \quad 0 < \alpha < 1,$$

$$3) \kappa(t) = h_{1-\alpha+\beta}(t) + h_{1-\alpha}(t), \quad 0 < \beta < \alpha < 1, \quad k(t) = t^{\alpha-1} E_{\beta,\alpha}(-t^{\beta}),$$

where $E_{\beta,\alpha}$ stands for the two-parameters Mittag-Leffler function:

$$E_{\beta,\alpha}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\beta k + \alpha)}, \quad \beta > 0, \quad \alpha, z \in \mathbb{C}.$$

Examples of the Sonin kernels

4) $\kappa(t) = t^{\alpha-1} E_{\beta,\alpha}^{\gamma}(-\lambda t^{\beta})$, $k(t) = t^{-\alpha} E_{\beta,1-\alpha}^{-\gamma}(-\lambda t^{\beta})$,
 $0 < \alpha < 1$, $\beta > 0$, $\lambda \in \mathbb{R}$, where $E_{\beta,\alpha}^{\gamma}$ is the three-parameters
 Mittag-Leffler function or the Prabhakar function:

$$E_{\beta,\alpha}^{\gamma}(z) := \sum_{n=0}^{+\infty} \frac{(\gamma)_n}{n! \Gamma(\beta n + \alpha)}, \quad z, \alpha, \gamma \in \mathbb{C}, \quad \beta > 0.$$

More examples and methods for derivation of the new Sonin kernels:

S.G. Samko, R.P. Cardoso: Integral equations of the first kind of Sonine type. Int. J. Math. Sci. 57(2003), 3609-3632.

Yu. Luchko: On Symmetrical Sonin Kernels in Terms of Hypergeometric-Type Functions. Mathematics 2024, 12(24), 3943.

Kochubei class of the Sonin kernels

A.N. Kochubei: General fractional calculus, evolution equations, and renewal processes. *Integr. Equa. Oper. Theory* **2011**, 71, 583–600

Kochubei class $\mathcal{K}$ of the Sonin kernels:

- (K1) The Laplace transform $\tilde{k}$ of k , exists for all $p > 0$;
- (K2) $\tilde{k}(p)$ is a Stieltjes function (Stieltjes functions are a subclass of completely monotone functions such that their reciprocals are complete Bernstein functions);
- (K3) $\tilde{k}(p) \rightarrow 0$ and $p\tilde{k}(p) \rightarrow +\infty$ as $p \rightarrow +\infty$;
- (K4) $\tilde{k}(p) \rightarrow +\infty$ and $p\tilde{k}(p) \rightarrow 0$ as $p \rightarrow 0$.

GFI and GFD with the Sonin kernels from the class $\mathcal{K}$

Kochubei introduced the regularized GFD (GFD in the Caputo sense) with the kernels from $\mathcal{K}$:

$$(*\mathbb{D}_{(k)} f)(t) := (\mathbb{D}_{(k)} [f(\cdot) - f(0)])(t) = (\mathbb{I}_{(k)} f')(t), \quad t > 0,$$

For the GFDs with the kernels from $\mathcal{K}$, Kochubei derived the following results:

- **1st Fundamental Theorem of FC** ($*\mathbb{D}_{(k)}$ is a left inverse operator to the GFI on the suitable spaces of functions).
- **Fractional relaxation equation with $*\mathbb{D}_{(k)}$** $\Rightarrow$ solution to the Cauchy problem is completely monotone.
- **Cauchy problem for the general time-fractional diffusion equation with $*\mathbb{D}_{(k)}$** $\Rightarrow$ The fundamental solution is a spatial probability density function evolving in time.

Further results for the PDEs with the GFDs

Yu. Luchko, M. Yamamoto: General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.* 19 (2016), 676-695

- Estimates for the GFD at the maximum points of the functions (**Analogies of the Fermat theorem for the general fractional derivatives**)
- Maximum principle for the initial-boundary-value problems (**my talk at the conference GPCO2026**)
- ...

These results were derived for the GFDs with the Sonin kernels from a class different from the Kochubei class.

Analogies of the Fermat theorem for the general fractional derivatives

Let the conditions

$$\text{L1)} \quad k \in C^1(\mathbb{R}_+) \cap L_1^{loc}(\mathbb{R}_+),$$

$$\text{L2)} \quad k(\tau) > 0 \text{ and } k'(\tau) < 0 \text{ for } \tau > 0,$$

$$\text{L3)} \quad k(\tau) = o(\tau^{-1}), \quad \tau \rightarrow 0.$$

be fulfilled.

Let a function $f \in C([0, T])$ attain its maximum over the interval $[0, T]$ at the point t_0 , $t_0 \in (0, T]$, and $f' \in C(0, T] \cap L_1(0, T)$. Then the following inequalities hold true:

$$(\mathbb{D}_{(k)} f)(t_0) \geq k(t_0)f(t_0),$$

$$(*\mathbb{D}_{(k)} f)(t_0) \geq k(t_0)(f(t_0) - f(0)) \geq 0.$$

Sketch of the proof

Under the conditions posed on the functions f and k , the general fractional derivative $\mathbb{D}_{(k)}$ can be represented as follows:

$$\begin{aligned}
 (\mathbb{D}_{(k)} f)(t) &= \frac{d}{dt} \int_0^t k(t-\tau) f(\tau) d\tau \\
 &= \frac{d}{dt} \int_0^t k(t-\tau) (f(\tau) - f(t) + f(t)) d\tau \\
 &= \frac{d}{dt} \int_0^t k(t-\tau) (f(\tau) - f(t)) d\tau + f'(t) \int_0^t k(t-\tau) d\tau + k(t)f(t) \\
 &= I(t) + f'(t) \int_0^t k(t-\tau) d\tau + k(t)f(t).
 \end{aligned}$$

Sketch of the proof

The integral

$$I(t) = \frac{d}{dt} \int_0^t k(t-\tau)(f(\tau) - f(t)) d\tau, \quad t \in (0, T]$$

exists for all $t \in (0, T]$ because of the conditions posed on the functions f and k .

Due to the inclusion $f' \in C(0, T]$ and the condition L3), for any t , $t > 0$ and τ , $0 < \tau < t$ we first get the representation

$f(\tau) - f(t) = -f'(\zeta)(t - \tau)$, $\zeta = \zeta(t, \tau) \in [\tau, t]$ and then the formula

$$\lim_{\tau \rightarrow t} k(t - \tau)(f(\tau) - f(t)) = -\lim_{\tau \rightarrow t} k(t - \tau)f'(\zeta)(t - \tau) = 0.$$

Thus we can represent the integral I as follows:

$$I(t) = \int_0^t k'(t - \tau)(f(\tau) - f(t)) d\tau - f'(t) \int_0^t k(t - \tau) d\tau.$$

Sketch of the proof

Using the last formula for the integral $I(t)$, we arrive at the following representation

$$(\mathbb{D}_{(k)} f)(t) = \int_0^t k'(t - \tau)(f(\tau) - f(t)) d\tau + k(t)f(t)$$

that is valid for all $t \in (0, T]$. Let us now substitute the point $t = t_0$, where the function f attains its maximum into the last formula and take into consideration the condition L2). We thus obtain the estimate

$$(\mathbb{D}_{(k)} f)(t_0) = \int_0^{t_0} k'(t_0 - \tau)(f(\tau) - f(t_0)) d\tau + k(t_0)f(t_0) \geq k(t_0)f(t_0).$$

For the regularized general fractional derivative, we readily arrive at the estimate

$$(*\mathbb{D}_{(k)} f)(t_0) = (\mathbb{D}_{(k)} f)(t_0) - k(t_0)f(0) \geq k(t_0)f(t_0) - k(t_0)f(0) \geq 0$$

and the proof is completed.

Particular cases

For the Riemann-Liouville and Caputo fractional derivatives (the case of the kernel $k(t) = h_{1-\alpha}(t)$), the derived inequalities take the form

$$(D_{RL}^{\alpha} f)(t_0) = \frac{d}{dt} (I_{0+}^{1-\alpha} f)(t)|_{t=t_0} \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} f(t_0)$$

and

$$(D_C^{\alpha} f)(t_0) = (I_{0+}^{1-\alpha} \frac{d}{dt} f)(t)|_{t=t_0} \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} (f(t_0) - f(0)) \geq 0.$$

Hilfer fractional derivative

The Hilfer fractional derivative of the order α , $0 < \alpha < 1$ and type β , $0 \leq \beta \leq 1$ is defined by the formula

$$(D_{0+}^{\alpha,\beta} f)(t) = (I_{0+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0+}^{(1-\alpha)(1-\beta)} f)(t), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.$$

The Riemann-Liouville and the Caputo fractional derivatives are particular cases of the Hilfer derivative with $\beta = 0$ and $\beta = 1$, respectively:

$$(D_{0+}^{\alpha,0} f)(x) = (I_{0+}^0 \frac{d}{dt} I_{0+}^{1-\alpha} f)(t) = \frac{d}{dt} (I_{0+}^{1-\alpha} f)(t) = (D_{RL}^{\alpha} f)(t),$$

$$(D_{0+}^{\alpha,1} f)(t) = (I_{0+}^{1-\alpha} \frac{d}{dt} I_{0+}^0 f)(t) = (I_{0+}^{1-\alpha} \frac{d}{dt} f)(t) = (D_C^{\alpha} f)(t).$$

Hilfer fractional derivative

Yu. Luchko: Maximum principle and its application for the time-fractional diffusion equations. Fract. Calc. Appl. Anal. 14(2011), 110-124.

The Hilfer fractional derivative of a function f at its maximum point t_0 is always non-negative for any α , $0 < \alpha < 1$, i.e.,

$$(D^{\alpha,\beta} f)(t_0) \geq 0, \quad 0 < \alpha < 1$$

only if $\beta = 1$, i.e. only in the case of the Caputo fractional derivative.

Example

We consider a family of functions

$$f(t) := -at^2 + bt + c, \quad 0 < a, \quad 0 < b \leq 2a, \quad c \in \mathbb{R}$$

on the closed interval $[0, 1]$. The conditions on the parameters of the function f ensure the existence of the maximum point $t = b/(2a)$ that belongs to the interval $[0, 1]$.

For $0 < \alpha < 1$ and $0 \leq \beta < 1$, we get

$$(D^{\alpha, \beta} t^\gamma)(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} t^{\gamma - \alpha}$$

and then

$$(D^{\alpha, \beta} f)(t) = -\frac{2at^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{bt^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

Example

Thus, for $0 < \alpha < 1$ and $0 \leq \beta < 1$ we get

$$(D_t^{\alpha,\beta} f)(t) \Big|_{t=\frac{b}{2a}} = \left(\frac{b}{2a}\right)^{-\alpha} \left(\frac{b^2(1-\alpha)}{2a\Gamma(3-\alpha)} + \frac{c}{\Gamma(1-\alpha)} \right).$$

Because the coefficient c can be chosen to be both positive and negative and with an arbitrary absolute value, the fractional derivative $D^{\alpha,\beta} f$ can be evidently made both positive and negative and cannot be hold nonnegative in any maximum point of any function f .

Remark: the formulas presented above are valid only for the values $0 \leq \beta < 1$. In the case $\beta = 1$, i.e. for the Caputo fractional derivative, we get another formula

$$(D_t^{\alpha,1} f)(t) \Big|_{t=\frac{b}{2a}} = \left(\frac{b}{2a}\right)^{-\alpha} \left(\frac{b^2(1-\alpha)}{2a\Gamma(3-\alpha)} \right)$$

because the Caputo derivative of a constant function is equal to zero:

$$(D_C^{\alpha,1} t^0)(t) \equiv 0, \quad \forall t \geq 0.$$

Part 4: The general Fractional Calculus operators with the Sonin kernels: Properties and Applications

Yuri Luchko

Department of Mathematics, Physics, and Chemistry
Berlin University of Applied Sciences and Technology
Berlin, Germany

Workshop on Optimization Methods and Applications
Bialystok University of Technology
Bialystok, Poland
September 19 - 20, 2026

Outline of the talk:

- General fractional integrals and derivatives with the Sonin kernels from the class $\mathcal{S}_{-1}$
- Generalized convolution Taylor formula
- Convolution series and the generalized convolution Taylor series
- References and further results

Sonin kernels from the class $\mathcal{S}_{-1}$

In this talk, we consider a general FC for the GFIs and GFDs with the kernels from the **class $\mathcal{S}_{-1}$ of the Sonin kernels** defined as follows:

$$(\kappa, k \in \mathcal{S}_{-1}) \Leftrightarrow ((\kappa * k)(t) = 1) \wedge (\kappa, k \in C_{-1,0}(0, +\infty)),$$

where

$$C_{-1,0}(0, +\infty) = \{f : f(t) = t^p f_1(t), -1 < p < 0, f_1 \in C[0, +\infty), f_1(0) \neq 0\}$$

The kernels from $\mathcal{S}_{-1}$ possess integrable singularities of the power law type at the origin. **There exist Sonin kernels with other types of singularities!**

The theory will be constructed on the **space of functions $C_{-1}(0, +\infty)$ and its subspaces:**

$$C_{-1}(0, +\infty) = \{f : f(t) = t^p f_1(t), p > -1, f_1 \in C[0, +\infty)\}.$$

GFI and GFD with the kernels from $\mathcal{S}_{-1}$

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associated Sonin kernel. The GFI with the kernel κ is defined by

$$(\mathbb{I}_{(\kappa)} f)(t) = (\kappa * f)(t) = \int_0^t \kappa(t - \tau) f(\tau) d\tau.$$

The GFD and the regularized GFD with kernel k :

$$(\mathbb{D}_{(k)} f)(t) = \frac{d}{dt} (k * f)(t) = \frac{d}{dt} (\mathbb{I}_{(k)} f)(t), \quad t > 0,$$

$$(*\mathbb{D}_{(k)} f)(t) = (\mathbb{D}_{(k)} [f(\cdot) - f(0)])(t) = \\ (\mathbb{D}_{(k)} f)(t) - f(0)k(t), \quad t > 0.$$

Basic properties of the GFI with the kernels from $\mathcal{S}_{-1}$

Mapping property:

$$\mathbb{I}_{(\kappa)} : C_{-1}(0, +\infty) \rightarrow C_{-1}(0, +\infty), \quad \kappa \in \mathcal{S}_{-1}.$$

Commutativity law:

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)} \mathbb{I}_{(\kappa_1)}, \quad \kappa_1, \kappa_2 \in \mathcal{S}_{-1},$$

Index law:

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)}, \quad \kappa_1, \kappa_2 \in \mathcal{S}_{-1}.$$

Example: Index law for the Riemann-Liouville fractional integrals

$$I_{0+}^{\alpha} I_{0+}^{\beta} = I_{0+}^{\alpha+\beta}$$

because of the relation

$$(h_{\alpha} * h_{\beta})(t) = h_{\alpha+\beta}(t), \quad h_{\gamma}(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \quad \gamma > 0.$$

n -fold GFI

Let $\kappa \in \mathcal{S}_{-1}$. The n -fold GFI ($n \in \mathbb{N}$) is defined as a composition of n GFIs with the kernel κ :

$$(\mathbb{I}_{(\kappa)}^{<n>} f)(t) := \underbrace{(\mathbb{I}_{(\kappa)} \dots \mathbb{I}_{(\kappa)})}_{n \text{ times}} f(t), \quad t > 0.$$

By employing the index law, we can represent the n -fold GFI as a GFI with the kernel $\kappa^{<n>}$:

$$(\mathbb{I}_{(\kappa)}^{<n>} f)(t) = (\kappa^{<n>} * f)(t), \quad t > 0,$$

where the expression $\kappa^{<n>}$ stands for a convolution power

$$\kappa^{<n>}(t) := \begin{cases} \kappa(t), & n = 1, \\ \underbrace{(\kappa * \dots * \kappa)}_{n \text{ times}}(t), & n = 2, 3, \dots \end{cases}$$

n -fold sequential GFDs

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associated Sonin kernel. The n -fold sequential GFD and the n -fold sequential regularized GFD are defined as follows, respectively:

$$(\mathbb{D}_{(k)}^{<n>} f)(t) := \underbrace{(\mathbb{D}_{(k)} \dots \mathbb{D}_{(k)} f)(t)}_{n \text{ times}}, \quad t > 0,$$

$$(*\mathbb{D}_{(k)}^{<n>} f)(t) := \underbrace{(*\mathbb{D}_{(k)} \dots *\mathbb{D}_{(k)} f)(t)}_{n \text{ times}}, \quad t > 0.$$

1st Fundamental Theorem of FC for the n -fold sequential GFD and the n -fold sequential regularized GFD

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associated Sonin kernel.

Then, the n -fold sequential GFD is a left inverse operator to the n -fold GFI on the space $C_{-1}(0, +\infty)$:

$$(\mathbb{D}_{(k)}^{<n>} \mathbb{I}_{(\kappa)}^{<n>} f)(t) = f(t), \quad f \in C_{-1}(0, +\infty), \quad t > 0$$

and the n -fold sequential regularized GFD is a left inverse operator to the n -fold GFI on the space $\mathbb{I}_{(k)}(C_{-1}(0, +\infty))$:

$$(*\mathbb{D}_{(k)}^{<n>} \mathbb{I}_{(\kappa)}^{<n>} f)(t) = f(t), \quad f \in \mathbb{I}_{(k)}(C_{-1}(0, +\infty)), \quad t > 0,$$

where $\mathbb{I}_{(k)}(C_{-1}(0, +\infty)) := \{f : f(t) = (\mathbb{I}_{(k)} \phi)(t), \phi \in C_{-1}(0, +\infty)\}$.

2nd Fundamental Theorem of FC for the sequential GFD

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associated Sonin kernel.

For a function $f \in C_{-1,(k)}^n(0, +\infty)$, the formula

$$(\mathbb{I}_{(\kappa)}^{<n>} \mathbb{D}_{(k)}^{<n>} f)(t) = f(t) - \sum_{j=0}^{n-1} \left(k * \mathbb{D}_{(k)}^{<j>} f \right) (0) \kappa^{<j+1>}(t) =$$

$$f(t) - \sum_{j=0}^{n-1} \left(\mathbb{I}_{(k)} \mathbb{D}_{(k)}^{<j>} f \right) (0) \kappa^{<j+1>}(t), \quad t > 0$$

holds valid, where $\mathbb{I}_{(\kappa)}^{<n>}$ is the n -fold GFI, $\mathbb{D}_{(k)}^{<n>}$ is the n -fold sequential GFD, and $C_{-1,(k)}^n(0, +\infty) = \{f \in C_{-1}(0, +\infty) : (\mathbb{D}_{(k)}^{<j>} f) \in C_{-1}(0, +\infty), j = 1, \dots, n\}$.

Generalized convolution Taylor formula for the GFD

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associated Sonin kernel.

For a function $f \in C_{-1,(k)}^n(0, +\infty)$, the generalized convolution Taylor formula

$$f(t) = \sum_{j=0}^{n-1} a_j \kappa^{<j+1>}(t) + r_n(t), \quad t > 0$$

holds valid. The coefficients a_j are given by the expression

$$a_j = \left(k * \mathbb{D}_{(k)}^{<j>} f \right) (0) = \left(\mathbb{I}_{(k)} \mathbb{D}_{(k)}^{<j>} f \right) (0), \quad j = 0, 1, \dots, n-1$$

and the remainder can be represented as follows:

$$r_n(t) = (\mathbb{I}_{(\kappa)}^{<n>} \mathbb{D}_{(k)}^{<n>} f)(t) = (\mathbb{D}_{(k)}^{<n>} f)(\tau) (\{1\} * \kappa^{<n>})(t), \quad 0 < \tau \leq t,$$

where $\mathbb{I}_{(\kappa)}^{<n>}$ is the n -fold GFI and $\mathbb{D}_{(k)}^{<n>}$ is the n -fold sequential GFD.

Generalized convolution Taylor formula for the R-L derivative

In the case $\kappa(t) = h_\alpha(t)$, $0 < \alpha < 1$ (the kernel of the Riemann-Liouville fractional integral), the generalized convolution Taylor formula is reduced to the following known form:

$$f(t) = t^{\alpha-1} \sum_{j=0}^{n-1} a_j \frac{t^{\alpha j}}{\Gamma(\alpha j + \alpha)} + r_n(t), \quad t > 0,$$

$$a_j = \left(I_{0+}^{1-\alpha} (D_{0+}^\alpha)^{<j>} f \right) (0), \quad j = 0, 1, \dots, n-1$$

with the remainder

$$r_n(t) = \left(I_{0+}^{n\alpha} (D_{0+}^\alpha)^{<n>} f \right) (t) = \left((D_{0+}^\alpha)^{<n>} f \right) (\tau) \frac{t^{\alpha n}}{\Gamma(\alpha n + \alpha)}, \quad 0 < \tau \leq t,$$

where I_{0+}^α is the Riemann-Liouville fractional integral and $(D_{0+}^\alpha)^{<n>}$ is the n -fold sequential Riemann-Liouville fractional derivative. The conventional Taylor formula is obtained by letting α go to 1.

Convolution series

Let a function $\kappa \in C_{-1}(0, +\infty)$ be represented in the form

$$\kappa(t) = h_p(t) \kappa_1(t), \quad t > 0, \quad p > 0, \quad \kappa_1 \in C[0, +\infty)$$

and the convergence radius of the power series

$$\Sigma(z) = \sum_{j=0}^{+\infty} a_j z^j, \quad a_j \in \mathbb{C}, \quad z \in \mathbb{C}$$

be non-zero. Then the convolution series

$$\Sigma_{\kappa}(t) = \sum_{j=0}^{+\infty} a_j \kappa^{<j+1>}(t)$$

is convergent for all $t > 0$ and $\Sigma_{\kappa} \in C_{-1}(0, +\infty)$. Moreover, the series

$$t^{1-\alpha} \Sigma_{\kappa}(t) = \sum_{j=0}^{+\infty} a_j t^{1-\alpha} \kappa^{<j+1>}(t), \quad \alpha = \min\{p, 1\}$$

is uniformly convergent for $t \in [0, T]$ for any $T > 0$.

Convolution series

If we start with the geometric series

$$\Sigma(z) = \sum_{j=0}^{+\infty} \lambda^j z^j, \quad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}$$

that for $\lambda \neq 0$ has the convergence radius $r = 1/|\lambda|$, then the convolution series ($\kappa \in C_{-1}(0, +\infty)$)

$$I_{\kappa, \lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j \kappa^{<j+1>}(t), \quad \lambda \in \mathbb{C}$$

is convergent for all $t > 0$ and defines a function from the space $C_{-1}(0, +\infty)$.

The function $I_{\kappa, \lambda}(t)$ plays an important role while solving the linear fractional differential equations with the GFDs.

Convolution series

For the kernel function $\kappa = \{1\}$, we immediately get the formula $\kappa^{<j+1>}(t) = \{1\}^{<j+1>}(t) = h_{j+1}(t)$. Then the convolution series $l_{\kappa,\lambda}(t)$ becomes a familiar power series for the exponential function:

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j h_{j+1}(t) = \sum_{j=0}^{+\infty} \frac{(\lambda t)^j}{j!} = e^{\lambda t}.$$

In the case of the kernel $\kappa(t) = h_{\alpha}(t)$ of the Riemann-Liouville fractional integral, the formula $\kappa^{<j+1>}(t) = h_{\alpha}^{<j+1>}(t) = h_{(j+1)\alpha}(t)$ is valid and the convolution series $l_{\kappa,\lambda}(t)$ takes the form

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j h_{(j+1)\alpha}(t) = t^{\alpha-1} \sum_{j=0}^{+\infty} \frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha + \alpha)} = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}).$$

Convolution series

Now we proceed with analysis of the convolution series generated by the kernel functions $\kappa \in \mathcal{S}_{-1}$.

In what follows, we always assume that the convergence radius of the power series $\Sigma(z) = \sum_{j=0}^{+\infty} a_j z^j$, $a_j \in \mathbb{C}$, $z \in \mathbb{C}$ is non-zero. Then the convolution series defines a function from the space $C_{-1}(0, +\infty)$ that we denote by f :

$$f(t) = \sum_{j=0}^{+\infty} a_j \kappa^{<j+1>}(t).$$

Problem: determine the coefficients a_j , $j \in \mathbb{N}_0$ in terms of the function f .

Generalized convolution Taylor series

Yu. Luchko: Convolution series and the generalized convolution Taylor formula. Fract. Calc. Appl. Anal. 25 (2022), 207-228

Any function f in form of a convolution series generated by the Sonin kernel $\kappa \in \mathcal{S}_{-1}$ can be represented as the following generalized convolution Taylor series:

$$f(t) = \sum_{j=0}^{+\infty} \left(\mathbb{I}_{(k)} \mathbb{D}_{(k)}^{<j>} f \right) (0) \kappa^{<j+1>}(t),$$

where $\mathbb{I}_{(k)}$ is the GFI with the kernel k that is the Sonin kernel associated to the kernel κ and $\mathbb{D}_{(k)}^{<j>}$ is the j -fold sequential GFD.

Hilfer fractional derivative

The Hilfer fractional derivative of the order α , $0 < \alpha < 1$ and type β , $0 \leq \beta \leq 1$ is defined by the formula

$$(D_{0+}^{\alpha,\beta} f)(t) = (I_{0+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0+}^{(1-\alpha)(1-\beta)} f)(t), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.$$

The Riemann-Liouville and the Caputo fractional derivatives are particular cases of the Hilfer derivative with $\beta = 0$ and $\beta = 1$, respectively:

$$(D_{0+}^{\alpha,0} f)(x) = (I_{0+}^0 \frac{d}{dt} I_{0+}^{1-\alpha} f)(t) = \frac{d}{dt} (I_{0+}^{1-\alpha} f)(t) = (D_{RL}^{\alpha} f)(t),$$

$$(D_{0+}^{\alpha,1} f)(t) = (I_{0+}^{1-\alpha} \frac{d}{dt} I_{0+}^0 f)(t) = (I_{0+}^{1-\alpha} \frac{d}{dt} f)(t) = (D_C^{\alpha} f)(t).$$

1st level kernels of the GFD

The functions $\kappa, k_1, k_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfy the condition

$$(\kappa * k_1 * k_2)(t) = h_1(t) = 1, t > 0$$

are called the 1st level kernels of the GFDs.

The set of the 1st level kernels that are from the space $C_{-1}(0, +\infty)$ is denoted by $\mathcal{L}_1^1$ (1st level kernels of the GFDs of the order less than one).

Example: The power law functions (kernels of the Riemann-Liouville fractional integral and the Hilfer fractional derivative)

$$\kappa(t) = h_\alpha(t), k_1(t) = h_\gamma(t), k_2(t) = h_{1-\alpha-\gamma}(t), t > 0$$

whose parameters satisfy the conditions

$$0 < \alpha < 1, 0 < \gamma < 1 - \alpha.$$

Under these conditions they are from the class $\mathcal{L}_1^1$ due to the formula

$$(h_\alpha * h_\beta)(t) = h_{\alpha+\beta}(t), t > 0, \alpha > 0, \beta > 0.$$

1st level GFD

Let $(\kappa, k_1, k_2) \in \mathcal{L}_1^1$.

The 1st level GFD with the kernels (k_1, k_2) is given by the expression

$$({}_1L\mathbb{D}_{(k_1, k_2)} f)(t) := \left(\mathbb{I}_{(k_1)} \frac{d}{dt} \mathbb{I}_{(k_2)} f \right) (t),$$

where $\mathbb{I}_{(k_1)}$ and $\mathbb{I}_{(k_2)}$ are the GFIs.

Two important particular cases of the 1st level GFD are the GFD of the Riemann-Liouville type and the regularized GFD:

$$({}_1L\mathbb{D}_{(h_0, k_2)} f)(t) = \left(\mathbb{I}_{(h_0)} \frac{d}{dt} \mathbb{I}_{(k_2)} f \right) (t) = \left(\frac{d}{dt} \mathbb{I}_{(k_2)} f \right) (t) = (\mathbb{D}_{(k_2)} f)(t),$$

$$({}_1L\mathbb{D}_{(k_1, h_0)} f)(t) = \left(\mathbb{I}_{(k_1)} \frac{d}{dt} \mathbb{I}_{(h_0)} f \right) (t) = (\mathbb{I}_{(k_1)} f') (t) = ({}_*\mathbb{D}_{(k_1)} f)(t).$$

For the properties of the 1st level GFDs see

Luchko, Yu. The 1st Level General Fractional Derivatives and some of their Properties. J Math Sci, 266 (2022), 709-722.

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Thank you very much for your attention!

$$({}_*\mathbb{D}_{(k)} f)(t) := \left(\mathbb{D}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right) (t), \quad t > 0$$

$$f(t) = \sum_{j=0}^{+\infty} \left(\mathbb{I}_{(k)} \mathbb{D}_{(k)}^{<j>} f \right) (0) \kappa^{<j+1>}(t)$$

Questions and comments are welcome!