

VARIATIONAL ANALYSIS AND APPLICATIONS

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THE FUNDAMENTAL VARIATIONAL PRINCIPLE

Namely, because the shape of the whole universe is the most perfect and, in fact, designed by the wisest creator, nothing in all the world will occur in which no maximum or minimum rule is somehow shining forth...

Leonhard Euler (1744)

INTRINSIC NONSMOOTHNESS

is typically encountered in applications of modern variational principles and techniques to numerous problems arising in pure and applied mathematics particularly in analysis, geometry, dynamical systems, optimization, equilibrium, control, economics, ecology, computer science, data science, machine learning, AI...

REMARKABLE CLASSES OF NONSMOOTH FUNCTIONS

MARGINAL/VALUE FUNCTIONS

$$\mu(x) := \inf \left\{ \varphi(x, y) \mid y \in G(x) \right\}$$

crucial in perturbation and sensitivity analysis, stability, and many other issues. In particular, **DISTANCE FUNCTIONS**

$$\text{dist}(x; \Omega) := \inf \left\{ \|x - y\| \mid y \in \Omega \right\} \quad \text{or generally} \quad \rho(x, z) := \text{dist}(x; F(z))$$

naturally appear via variational principles and penalization

EKELAND'S VARIATIONAL PRINCIPLE

THEOREM. Let (X, d) be a complete metric space. Assume that $\varphi: X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Let $\varepsilon > 0$ and $x_0 \in X$ be given such that $\varphi(x_0) \leq \inf_X \varphi + \varepsilon$. Then for any $\lambda > 0$ there exists $\bar{x} \in X$ satisfying the conditions

- $\varphi(\bar{x}) \leq \varphi(x_0)$
- $d(\bar{x}, x_0) \leq \lambda$
- $\varphi(x) + (\varepsilon/\lambda)d(x, \bar{x}) > \varphi(\bar{x})$ for all $x \neq \bar{x}$.

EXERCISE: Prove the result in finite-dimensional spaces.

EXERCISE*: Prove the result in Banach spaces

INTRINSIC NONSMOOTHNESS (cont.)

MAXIMUM FUNCTIONS

$$f(x) = \max_{u \in U} g(x, u).$$

in particular, **HAMILTONIANS** in physics, mechanics, calculus of variations, systems control, variational inequalities, etc.

EXERCISE: Show that the maximum of convex functions is convex and give examples of nonconvex maximum functions.

NONSMOOTH/NONCONVEX SETS AND MAPPINGS

Parametric sets of feasible and optimal solutions in various problems of equilibrium, optimization, dynamics

Preference and production sets in economic modeling

Reachable sets in dynamical and control systems

Sets of Equilibria and **Equilibrium Constraints** in physical, mechanical, economic, ecological, and biological models

Regularizators in problems of machine learning

SUBDIFFERENTIALS

of $\varphi: \mathbb{R}^n \rightarrow \bar{\mathbb{R}} := (-\infty, \infty]$ with $\varphi(\bar{x}) < \infty$ should satisfy:

1) for convex functions φ reduces to

$$\partial^\bullet \varphi(\bar{x}) = \left\{ v \mid \varphi(x) - \varphi(\bar{x}) \geq \langle v, x - \bar{x} \rangle \text{ for all } x \in \mathbb{R}^n \right\}$$

2) If $\bar{x}$ is a local minimizer for φ , then $0 \in \partial^\bullet \varphi(\bar{x})$

3) Sum Rule (Basic Calculus)

$$\partial^\bullet (\varphi_1 + \varphi_2)(\bar{x}) \subset \partial^\bullet \varphi_1(\bar{x}) + \partial^\bullet \varphi_2(\bar{x})$$

4) Robustness

$$\partial^\bullet \varphi(\bar{x}) = \left\{ v \in \mathbb{R}^n \mid \begin{array}{l} \exists x_k \rightarrow \bar{x}, v_k \rightarrow v, \varphi(x_k) \rightarrow \varphi(\bar{x}), \\ \text{s.t. } v_k \in \partial^\bullet \varphi(x_k), \quad k = 1, 2, \dots \end{array} \right\}$$

EXERCISE: Prove that the properties 2) and 4) hold for the convex subdifferential

EXERCISE*: Prove that the property 3) holds for the convex subdifferential

THE BASIC SUBDIFFERENTIAL

of $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ at $\bar{x}$ is defined by the outer limit

$$\partial\varphi(\bar{x}) = \left\{ v \in \mathbb{R}^n \mid \begin{array}{l} \exists x_k \rightarrow \bar{x}, v_k \rightarrow v, \varphi(x_k) \rightarrow \varphi(\bar{x}), \\ \text{s.t. } v_k \in \hat{\partial}\varphi(x_k), \quad k = 1, 2, \dots \end{array} \right\}$$

where **regular/viscosity subdifferential** of φ at x is defined by

$$\hat{\partial}\varphi(x) := \left\{ v \mid \liminf_{u \rightarrow x} \frac{\varphi(u) - \varphi(x) - \langle v, u - x \rangle}{\|u - x\|} \geq 0 \right\}$$

The **basic subdifferential** is **minimal** among all subdifferentials satisfying 1)-4), nonempty

$\partial\varphi(\bar{x}) \neq \emptyset$ for Lipschitz functions

while often **nonconvex**, e.g., $\partial(-|x|)(0) = \{-1, 1\}$. Moreover, its **convexification**, made for convenience, can **dramatically worsen** the basic properties and applications.

EXERCISE: Show that for smooth functions, both subdifferentials reduce to the gradient

EXERCISE: Calculate the regular and basic subdifferentials of $\varphi(x) = |x|$ at $\bar{x} = 0$

EXERCISE: Calculate the regular and basic subdifferentials of $\varphi(x) = -|x|$ at $\bar{x} = 0$

EXERCISE: Calculate the regular and basic subdifferentials of $\varphi(x) = x^{1/3}$ at $\bar{x} = 0$

EXERCISE*: Calculate the above subdifferentials of $\varphi(x) = \|x\|$ and $\varphi(x) = \|\bar{x}\|$ at $\bar{x} = 0$ in $\mathbb{R}^2$ (or in $\mathbb{R}^n$)

VARIATIONAL GEOMETRY

The (basic, limiting) **NORMAL CONE** $N(\bar{x}; \Omega) := \partial\delta(\bar{x}; \Omega)$ to Ω at $\bar{x} \in \Omega$, where $\delta(x; \Omega) := 0$ for $x \in \Omega$ and $\delta(x; \Omega) := \infty$ for $x \notin \Omega$. It can be equivalently described as

$$N(\bar{x}; \Omega) = \left\{ v \in \mathbb{R}^n \left| \begin{array}{l} \exists x_k \rightarrow \bar{x}, \alpha_k \geq 0, w_k \in \Pi(x_k; \Omega) \\ \text{s.t. } \alpha_k(x_k - w_k) \rightarrow v \text{ as } k \rightarrow \infty \end{array} \right. \right\},$$

where $\Pi(x; \Omega)$ is the Euclidean projector of the set Ω , i.e.,

$$\Pi(x; \Omega) := \left\{ y \in \Omega \left| \|y - x\| = \min_{u \in \Omega} \|u - x\| \right. \right\}$$

EXERCISE*: Prove the aforementioned equivalence.

EXERCISE: Calculate the normal cone to the epigraphical set $\text{epi } |x|$ at $(0, 0)$

EXERCISE: Calculate the normal cone to the epigraphical set $\text{epi } (-|x|)$ at $(0, 0)$

EXERCISE: Calculate the normal cone to the graphical set $\text{gph } |x|$ at $(0, 0)$

EXTREMALITY OF SET SYSTEMS

DEFINITION. A vector $\bar{x} \in \Omega_1 \cap \Omega_2$ is a **LOCAL EXTREMAL POINT** of the system of closed sets $\{\Omega_1, \Omega_2\}$ in $\mathbb{R}^n$ if there exists a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in \mathbb{R}^n$ with $\|a\| < \varepsilon$ satisfying

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset$$

EXAMPLES:

- boundary point of closed sets
- local solutions to constrained optimization, multiobjective optimization, and other optimization-related problems
- minimax solutions and equilibrium points
- Pareto-type allocations in economics
- stationary points in mechanical and ecological models, etc.

EXERCISE: Prove the local extremality of boundary points.

EXERCISE: Show that a local minimizer of the problem

$$\text{minimize } \varphi(x) \text{ subject to } x \in \Omega \subset \mathbb{R}^n$$

can be reduced to the extremal system of sets.

THE BASIC EXTREMAL PRINCIPLE

THEOREM. Let $\bar{x}$ be a **LOCAL EXTREMAL POINT** for the system of closed sets $\{\Omega_1, \Omega_2\}$ in $\mathbb{R}^n$. Then there exists a dual element $0 \neq v \in \mathbb{R}^n$ such that

$$v \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2))$$

This is a **VARIATIONAL** counterpart of the separation theorem for the case of nonconvex sets, which plays a fundamental role in variational analysis and its applications.

SOME APPLICATIONS: Full Calculus of Generalized Differentiation; Well-Posedness in Optimization and Optimality Conditions; Sensitivity Analysis, ODE and PDE Control, Economic and Mechanical Equilibria, Numerical Analysis...

LOCAL EXTREMALITY FOR MANY SETS

Let $\Omega_1, \dots, \Omega_s$ with $s \geq 2$ be nonempty subsets of $\mathbb{R}^n$, and let $\bar{x}$ be their common point. We say that $\bar{x}$ is a **locally extremal point** of $\{\Omega_1, \dots, \Omega_s\}$ if there are sequences $\{a_{ik}\} \subset \mathbb{R}^n$, $i = 1, \dots, s$, and a neighborhood U of $\bar{x}$ such that $a_{ik} \rightarrow 0$ as $k \rightarrow \infty$ and

$$\bigcap_{i=1}^s \left(\Omega_i - a_{ik} \right) \cap U = \emptyset, \quad k \in \mathbb{N}.$$

In this case we say that $\{\Omega_1, \dots, \Omega_s, \bar{x}\}$ is an **extremal system**.

EXERCISE: Show that for $s = 2$, this reduces to the previous definition.

EXTREMAL PRINCIPLE FOR MANY SETS

THEOREM. Let $\{\Omega_1, \dots, \Omega_s, \bar{x}\}$ with $s \geq 2$ be an extremal system of closed sets in $\mathbb{R}^n$. Then there exist basic normals

$$v_i \in N(\bar{x}; \Omega_i), \quad i = 1, \dots, s,$$

to the sets Ω_i at the locally extremal point $\bar{x}$ such that

$$v_1 + \dots + v_s = 0, \quad \|v_1\|^2 + \dots + \|v_s\|^2 = 1.$$

PROOF. Without loss of generality, suppose that $U = \mathbb{R}^n$. Take $\{a_{ik}\}$ from the definition and for each $k = 1, 2, \dots$ consider the following problem of **unconstrained minimization**

$$\text{minimize } d_k(x) := \left[\sum_{i=1}^s \text{dist}^2(x + a_{ik}; \Omega_i) \right]^{1/2} + \|x - \bar{x}\|^2.$$

By the Weierstrass existence theorem, d_k admits an optimal solution x_k . The extremality of $\bar{x}$ yields

$$\alpha_k := \left[\sum_{i=1}^s \text{dist}^2(x_k + a_{ik}; \Omega_i) \right]^{1/2} > 0.$$

The optimality of x_k ensures that

$$d_k(x_k) = \alpha_k + \|x_k - \bar{x}\|^2 \leq \left[\sum_{i=1}^s \|a_{ik}\|^2 \right]^{1/2} \downarrow 0$$

and therefore $x_k \rightarrow \bar{x}$, $\alpha_k \downarrow 0$. Now for each $i = 1, \dots, s$, pick an Euclidean projection $w_{ik} \in \Pi(x_k + a_{ik}; \Omega_i)$ and consider yet another unconstrained optimization problem

$$\text{minimize } \rho_k(x) := \left[\sum_{i=1}^s \|x + a_{ik} - w_{ik}\|^2 \right]^{1/2} + \|x - \bar{x}\|^2$$

which has the same optimal solution x_k . Since $\alpha_k > 0$ and the Euclidean norm $\|\cdot\|$ is smooth on $\mathbb{R}^n \setminus \{0\}$, the function

$\rho_k(x)$ is continuously differentiable around x_k , and so the latter problem belongs to **smooth** unconstrained minimization. Thus the classical **Fermat stationary rule** yields the equality

$$\nabla \rho_k(x_k) = \sum_{i=1}^s v_{ik} + 2(x_k - \bar{x}) = 0$$

where $v_{ik} = (x_k + a_{ik} - w_{ik})/\alpha_k$, $i = 1, \dots, s$, and

$$\|v_{1k}\|^2 + \dots + \|v_{sk}\|^2 = 1, \quad k \in \mathbb{N}.$$

By the **compactness** of the unit sphere in $\mathbb{R}^n$, there exist $v_i \in \mathbb{R}^n$, $i = 1, \dots, s$, **not all zero**, such that $v_{ik} \rightarrow v_i$. Passing to the limit as $k \rightarrow \infty$ gives us **abstract Euler equation**

$$v_1 + \dots + v_s = 0.$$

Finally, it follows from the constructions of basic normals that

$$v_i \in N(\bar{x}; \Omega_i), \quad i = 1, \dots, s.$$

EXERCISE: Reduce the proof of the extremal principle for finitely many sets to the extremal principle for two sets by using the induction argument.

EXERCISE*: By using the extremal principle, prove the fundamental subrule for the basic subdifferential

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x})$$

provided that $\varphi_1: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is **locally Lipschitzian** around $\bar{x}$ and $\varphi_2: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is **lower semicontinuous** around this point.

EXTREMAL PRINCIPLE IN INFINITE DIMENSIONS

The above proof essentially uses the specific feature of $\mathbb{R}^n$:

- compactness of the unit ball (in applying the Weierstrass existence theorem)
- smoothness of the Euclidean norm at nonzero points
- compactness of the unit sphere (for passing to the limit of generalized normals)

Thus the derivation of the extremal principle in infinite dimensions requires significant modifications. The major ones are:

- Employing the Ekeland variational principle in complete metric spaces ensuring the existence of optimal solutions in some perturbed problems with nondifferentiable objectives

- Considering a large class of **smooth Banach spaces**, which admit an equivalent norm Fréchet differentiable off the origin. This class includes all **reflexive** Banach spaces. Furthermore, by using the method of **separable reduction** we elevate the results to the beautiful class of **Asplund spaces**, which are Banach spaces where each separable subspace has a separable dual.
- Obtain first the **approximate extremal principle** in Asplund spaces at points nearby the local extremal one and then derive the **exact extremal principle** at the local extremal point like in finite dimensions. To furnish this, we need certain **sequential normal compactness** of all but one sets in question that replaces the compactness of the unit sphere in infinite dimensions.

EXERCISE*: Formulate and prove the approximate extremal principle for two sets in smooth Banach spaces.

MATHEMATICAL PROGRAMMING

Consider the nonsmooth NP problem:

$$\begin{aligned} \text{minimize } \varphi_0(x) \quad \text{subject to} \quad & \varphi_i(x) \leq 0, \quad i = 1, \dots, m \\ & \varphi_i(x) = 0, \quad i = m+1, \dots, m+r \\ & x \in \Omega \subset \mathbb{R}^n \end{aligned}$$

THEOREM (generalized Lagrange multipliers). Let φ_i be locally Lipschitzian and Ω be locally closed around an optimal solution $\bar{x}$. Then there are $(\lambda_0, \dots, \lambda_{m+r}) \neq 0$ satisfying

$$\begin{aligned} \lambda_i &\geq 0, \quad i = 0, \dots, m, \quad \lambda_i \varphi_i(\bar{x}) = 0, \quad i = 1, \dots, m, \\ 0 &\in \partial \left(\sum_{i=0}^{m+r} \lambda_i \varphi_i \right) (\bar{x}) + N(\bar{x}; \Omega) \end{aligned}$$

Moreover, $\lambda_0 \neq 0$ (**Normality**) under appropriate **Constraint Qualification Conditions**

PROOF Assuming without loss of generality that $\varphi_0(\bar{x}) = 0$, we check that $(\bar{x}, 0)$ is a **locally extremal point** of the following system of locally closed sets in the product space $\mathbb{R}^n \times \mathbb{R}^{m+r+1}$:

$$\Omega_i := \left\{ (x, \alpha_0, \dots, \alpha_{m+r}) \mid \alpha_i \geq \varphi_i(x) \right\}, \quad i = 0, \dots, m,$$

$$\Omega_i := \left\{ (x, \alpha_0, \dots, \alpha_{m+r}) \mid \alpha_i = \varphi_i(x) \right\}, \quad i = m+1, \dots, m+r,$$

$$\Omega_{m+r+1} := \Omega \times \{0\}.$$

EXERCISE: Give a detailed proof of the theorem.

DYNAMICAL SYSTEMS

governed by differential inclusions

$$\dot{x}(t) \in F(x(t), t), \quad t \in [a, b], \quad x(a) = x_0 \in \mathbb{R}^n$$

where $\dot{x}$ stands for the time derivative and where $F: \mathbb{R}^n \times [a, b] \Rightarrow \mathbb{R}^n$ is a set-valued mapping. This description is important for qualitative theory of dynamical system and numerous applications, e.g., to various economic, ecological, biological, financial systems, climate research...

In particular, this covers parameterized control systems with

$$\dot{x} = g(x, u, t), \quad u(\cdot) \in U(x, t)$$

where the control region $U(x, t)$ depends on time and state

DISCRETE APPROXIMATIONS

Euler's finite difference (for simplicity)

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \quad h \rightarrow 0$$

Consider the mesh as $N \rightarrow \infty$

$$t_j := a + jh_N, \quad j = 0, \dots, N, \quad t_0 = a, \quad t_N = b, \quad h_N = (b - a)/N$$

Discrete Inclusions

$$x_N(t_{j+1}) \in x_N(t_j) + h_N F(x_N(t_j), t_j)$$

with piecewise linear Euler broken lines

Various Well-Posedness, Convergence, and Stability Issues of Numerical and Qualitative Analysis in Finite-Dimensional and Infinite-Dimensional Spaces

OPTIMAL CONTROL OF DIFFERENTIAL INCLUSIONS

minimize the cost functional

$$J[x] = \varphi(x(b)) \quad \text{subject to}$$

$$\dot{x}(t) \in F(x(t), t) \quad \text{a.e.} \quad t \in [a, b], \quad x(a) = x_0$$

$$x(b) \in \Omega \subset \mathbb{R}^n$$

where $F: \mathbb{R}^n \Rightarrow \mathbb{R}^n$ is a Lipschitz continuous set-valued mapping, Ω is a closed set, φ is a l.s.c. function

This covers various open-loop and closed-loop control systems with ODE dynamics and hard control and state constraints

CODERIVATIVES OF MAPPINGS

Let $F: X \rightrightarrows Y$ be a set-valued mapping with $(\bar{x}, \bar{y}) \in \text{gph} F$. Then $D^*F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph} F)\}$$

is called the **coderivative** of F at $(\bar{x}, \bar{y})$.

If $F: X \rightarrow Y$ is **smooth** around $\bar{x}$, then

$$D^*F(\bar{x})(y^*) = \{\nabla F(\bar{x})^* y^*\} \quad \text{for all } y^* \in Y^*$$

i.e., the coderivative is a proper generalization of the classical **adjoint derivative**. If $F: X \rightarrow Y$ is single-valued and **locally Lipschitzian** around $\bar{x}$, then the **scalarization formula** holds:

$$D^*F(\bar{x})(y^*) = \partial \langle y^*, F \rangle(\bar{x})$$

ENJOY FULL CALCULUS based on the Extremal Principle

EXERCISE: Calculate the coderivative of $F(x) = |x|$ at $(0, 0)$

EXERCISE: Calculate the coderivative of $F(x) = -|x|$ at $(0, 0)$

EXERCISE: Calculate the coderivative of $F(x) = -|x|^\alpha$ at $(0, 0)$ for $\alpha \in (0, 1)$

EXERCISE*: Prove the coderivative representation for smooth mappings between finite-dimensional spaces

CHARACTERIZATION OF WELL-POSEDNESS

$F: X \Rightarrow Y$ is called **Lipschitz-like** around $(\bar{x}, \bar{y}) \in \text{gph } F$ if there are neighborhood U of $\bar{x}$, V of $\bar{y}$ and modulus $\ell \geq 0$ for which we have the inclusion

$$F(x) \cap V \subset F(u) + \ell \|x - u\| B \quad \text{for all } x, u \in U$$

THEOREM. F is Lipschitz-like around $(\bar{x}, \bar{y})$ if and only if

$$D^*F(\bar{x}, \bar{y})(0) = \{0\}$$

The exact bound of Lipschitz moduli ℓ is calculated by

$$\text{lip } F(\bar{x}, \bar{y}) = \|D^*F(\bar{x}, \bar{y})\|$$

EXERCISE: Obtain the consequence of the coderivative criterion to characterizing the classical local Lipschitz property of single-valued mappings and real-valued functions

EXTENDED EULER-LAGRANGE+MAXIMUM PRINCIPLE

THEOREM. Let $\bar{x}(\cdot)$ be an optimal solution to the control problem. Then one has

Euler-Lagrange inclusion

$$\dot{p}(t) \in \text{co } D^*F(\bar{x}(t), \dot{x}(t))(-p(t)) \quad \text{a.e.},$$

Weierstrass-Pontryagin maximum condition

$$\langle p(t), \dot{\bar{x}}(t) \rangle = \max_{v \in F(\bar{x}(t))} \langle p(t), v \rangle \quad \text{a.e.}$$

transversality condition

$$-p(b) \in \lambda \partial \varphi(\bar{x}(b)) + N(\bar{x}(b); \Omega)$$

with nontriviality condition $(\lambda, p(\cdot)) \neq 0$.

PROOF: DISCRETE APPROXIMATIONS

HAMILTONIAN CONDITION

THEOREM. Let the sets $F(x) \subset \mathbb{R}^n$ be convex. Then the extended Euler-Lagrange inclusion is equivalent to the extended Hamiltonian inclusion

$$\dot{p}(t) \in \text{co} \left\{ u \mid (-u, \dot{x}(t)) \in \partial H(\bar{x}(t), p(t)) \right\} \text{ a.e.}$$

in terms of the basic subdifferential of the (true) Hamiltonian

$$H(x, p, t) := \sup \left\{ \langle p, v \rangle \mid v \in F(x, t) \right\}$$

which is intrinsically nonsmooth

SEMILINEAR EVOLUTION INCLUSIONS AND PDEs

minimize $J[x] := \varphi(x(b))$ subject to
mild solutions to the semilinear evolution inclusion

$$\dot{x}(t) \in Ax(t) + F(x(t), t), \quad x(a) = x_0$$

with the endpoint constraints

$$x(b) \in \Omega \subset X$$

where A is an unbounded generator of the C_0 semigroup, i.e.

$$\begin{aligned} x(t) &= e^{A(t-a)}x_0 + \int_a^t e^{A(t-s)}v(s) ds, \quad t \in [a, b] \\ v(t) &\in F(x(t), t), \quad t \in [a, b] \end{aligned}$$

in the sense of Bochner integration.

Cover PDE systems with parabolic and hyperbolic dynamics

CONTROLLED SWEEPING PROCESS

described by: minimize

$$J[x] := \varphi(x(T)) + \int_0^T f(x(t), \dot{x}(t)) dt \quad \text{s.t.}$$

$$\dot{x}(t) \in -N(x(t); C(t)), \quad t \in [0, T], \quad x(0) = x_0$$

with the controlled moving set

$$C(t) = \{x \in \mathbb{R}^n \mid \langle u(t), x \rangle \leq b(t)\}, \quad \|u(t)\| = 1$$

Shape optimization governed by discontinuous differential inclusions. The method of discrete approximations does the job

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